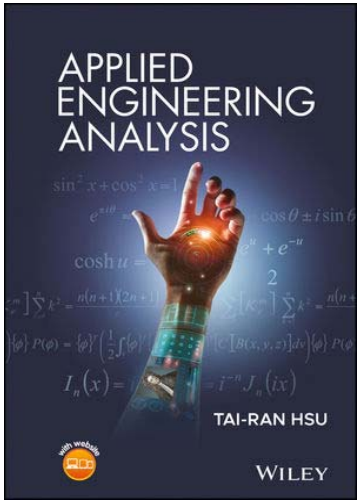
Applied Engineering Analysis
- slides for class teaching*

Chapter 8

Application of Second-order Differential Equations in Mechanical Engineering Analysis

* Based on the book of "Applied Engineering Analysis", by Tai-Ran Hsu, published by John Wiley & Sons, 2018 (ISBN 9781119071204)



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Chapter Learning Objectives

- **Refresh the solution methods for typical second-order homogeneous and non-homogeneous differential equations learned in previous math courses,**

- **Learn to derive homogeneous second-order differential equations for free vibration analysis of simple mass-spring system with and without damping effects,**

- **Learn to derive nonhomogeneous second-order differential equations for forced vibration analysis of simple mass-spring systems,**

- **Learn to use the solution of second-order nonhomogeneous differential equations to illustrate the resonant vibration of simple mass-spring systems and estimate the time for the rupture of the system under in resonant vibration,**

- **Learn to use the second order nonhomogeneous differential equation to predict the amplitudes of the vibrating mass in the situation of near-resonant vibration and the physical consequences to the mass-spring systems, and**

- **Learn the concept of modal analysis of machines and structures and the consequence of structural failure under the resonant and near-resonant vibration modes.**

Review Solution Method of Second Order, Homogeneous Ordinary Differential Equations

We will review the techniques available for solving typical second order differential equations at the beginning of this chapter.

The solution methods presented in the subsequent sections are generic and effective for engineering analysis.

8.2 Typical form of second-order homogeneous differential equations (p.243)

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \qquad (8.1)$$

where a and b are constants

The solution of Equation (8.1) u(x) may be obtained by ASSUMING:

$$\mathbf{u(x) = e^{mx}} \qquad (8.2)$$

in which **m** is a constant to be determined by the following procedure:

If the assumed solution u(x) in Equation (8.2) is a valid solution, it must SATISFY Equation (8.1). That is:

$$\frac{d^2\left(e^{mx}\right)}{dx^2} + a\frac{d\left(e^{mx}\right)}{dx} + b\left(e^{mx}\right) = 0 \qquad (a)$$

Because: $\frac{d^2\left(e^{mx}\right)}{dx^2} = m^2 e^{mx}$ and $\frac{d\left(e^{mx}\right)}{dx} = me^{mx}$

Substitution of the above expressions into Equation (a) will lead to:

$$m^2 e^{mx} + a\left(m e^{mx}\right) + b\left(e^{mx}\right) = 0$$

Because e^{mx} in the expression cannot be zero (why?), we thus have:

$$m^2 + am + b = 0 \qquad (8.3)$$

Equation (8.3) is a quadratic equation with unknown "m", and its 2 solutions for m are from:

$$m_1 = -\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4b} \qquad and \qquad m_2 = -\frac{a}{2} - \frac{1}{2}\sqrt{a^2 - 4b} \tag{8.4}$$

This leads to the following two possible solutions for the function u(x) in Equation (8.1):

$$u(x) = c_1\, e^{m_1 x} + c_2\, e^{m_2 x} \tag{8.5}$$

where c_1 and c_2 are the TWO arbitrary constants to be determined by TWO specified conditions, and m_1 and m_2 are expressed in Equation (8.4)

Because the constant coefficients a and b in Equation (8.1) are given in the differential equation, the values these constants a, b will result in significantly different forms in the solution as shown in Equation (8.5) due to the "square root" parts in the expression of m_1 and m_2 in Equation (8.4). Because <u>square root of negative numbers</u> will lead to a complex number in the solution of the differential equation, which requires a special way of expressing it.

We thus need to look into the following 3 possible cases involving relative magnitudes of the two coefficients a and b in Equation (8.1).

Case 1. $a^2 - 4b > 0$:

In such case, we realize that both m_1 and m_2 are real numbers. The solution of the Equation (8.1) is:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\sqrt{a^2-4b}\,x/2} + c_2 e^{-\sqrt{a^2-4b}\,x/2}\right) \qquad (8.6)$$

Case 2. $a^2 - 4b < 0$:

As described earlier, both these roots become complex numbers involving real and imaginary parts. The substitution of the m_1 and m_2 into Equation (8.5) will lead to the following:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\frac{ix}{2}\sqrt{4b-a^2}} + c_2 e^{-\frac{ix}{2}\sqrt{4b-a^2}}\right) \qquad (8.7)$$

in which, $i = \sqrt{-1}$. The complex form of the solution in Equation (8.7) is not always easily comprehended and manipulative in engineering analyses, a more commonly used form involving trigonometric functions are used instead:

$$u(x) = e^{-\frac{ax}{2}}\left[A\,Sin\left(\frac{1}{2}\sqrt{4b-a^2}\right)x + B\,Cos\left(\frac{1}{2}\sqrt{4b-a^2}\right)x\right] \qquad (8.8)$$

where A and B are arbitrary constants to be determined by given conditions.

The expression in Equation (8.8) may be derived from Equation (8.7) using the Biot relation that has the form: $e^{\pm i\theta} = Cos\theta \pm i\,Sin\theta$

For a special case with coefficient a = 0 and b is a negative number, the solution of Equation (8.1) becomes:

$$u(x) = c_1 \cosh\left(2\sqrt{b}\right)x + c_2 \sinh\left(2\sqrt{b}\right)x \qquad (8.9)$$

where c_1 and c_2 are arbitrary constants to be determined by given conditions.

Case 3. $a^2 - 4b = 0$:

Recall Equation (8.4):

$$m_1 = -\frac{a}{2} + \frac{1}{2}\sqrt{a^2 - 4b} \qquad and \qquad m_2 = -\frac{a}{2} - \frac{1}{2}\sqrt{a^2 - 4b}$$

The condition $a^2 - 4b = 0$ will thus lead to a situation of: $m_1 = m_2 = -a/2$ (b)

Substituting these m_1 and m_2 into Equation (8.5) will result in:

$$u(x) = (c_1 + c_2)e^{-\frac{a}{2}x} \quad or \quad u_1(x) = ce^{-\frac{a}{2}x}$$

with only ONE term with ONE constant in the solution, which cannot be a complete solution for a 2nd order differential equation in Equation (8.1).

We will have to find the "missing" solution of u(x) for a second-order differential equation in Equation (8.1) by following the procedure:

. Let us try the following additional assumed form of the solution u(x) :

$$u_2(x) = V(x)\, e^{mx} \qquad (8.10)$$

where V(x) is an assumed function of x, and it needs to be determined

The assumed second solution in Equation (8.10) must satisfy Equation (8.1)

The Differential equation: $$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \qquad (8.1)$$

The assumed <u>second</u> solution to Equation (8.1) is: $u_2(x) = V(x)\, e^{mx}$, which leads to the following equality:

$$\frac{d^2\left[V(x)e^{mx}\right]}{dx^2} + a\frac{d\left[V(x)e^{mx}\right]}{dx} + b\left[V(x)e^{mx}\right] = 0 \qquad (c)$$

One would find: $$\frac{d\left[V(x)e^{mx}\right]}{dx} = mV(x)e^{mx} + e^{mx}\frac{dV(x)}{dx}$$

and $$\frac{d^2\left[V(x)e^{mx}\right]}{dx^2} = m\left[mV(x)e^{mx} + e^{mx}\frac{dV(x)}{dx}\right] + me^{mx}\frac{dV(x)}{dx} + e^{mx}\frac{d^2V(x)}{dx^2}$$

After substituting the above expressions into Equation (c), we will get:

$$\frac{d^2V(x)}{dx^2} + (2m+a)\frac{dV(x)}{dx} + \left(m^2 + am + b\right)V(x) = 0 \qquad (8.11)$$

Since $m^2 + am + b = 0$ in Equation (8.3), and $m = m_1 = m_2 = -a/2$ in Equation (b), so both the 2nd and 3rd term in Equation (8.11) drop out. We thus only have the first term To consider in the following special form of a 2nd order differential equation:

$$\frac{d^2V(x)}{dx^2} = 0$$

The solution of the above differential equation is: $V(x) = x$ after 2 sequential integrations

The solution V(x) = x leads to the missing second solution of the differential equation in Equation (8.1)

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \qquad (8.1)$$

in Case 3 with $a^2 - 4b = 0$ as:

$$u_2(x) = V(x)e^{mx} = xe^{mx} = xe^{-\frac{ax}{2}}$$

The general solution of Equation (8.1) with a^2-4b=0 thus becomes:

$$u(x) = u_1(x) + u_2(x)$$

or

$$u(x) = c_1 e^{-\frac{ax}{2}} + c_2 x e^{-\frac{ax}{2}} = (c_1 + c_2 x)e^{-\frac{ax}{2}} \qquad (8.12)$$

where the two arbitrary constants c_1 and c_2 are determined by the two given conditions with Equation (8.1).

Summary on Solutions of 2nd Order Homogeneous Differential Equations

The equation:

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \qquad (8.1)$$

with TWO given conditions

The solutions:

Case 1: $a^2 - 4b > 0$:

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\sqrt{a^2-4b}\,x/2} + c_2 e^{-\sqrt{a^2-4b}\,x/2}\right) \qquad (8.6)$$

Case 2: $a^2 - 4b < 0$:

$$u(x) = e^{-\frac{ax}{2}}\left[A\,Sin\left(\frac{1}{2}\sqrt{4b-a^2}\right)x + B\,Cos\left(\frac{1}{2}\sqrt{4b-a^2}\right)x\right] \qquad (8.8)$$

Case 3: $a^2 - 4b = 0$: —— A special case

$$u(x) = c_1 e^{-\frac{ax}{2}} + c_2 x e^{-\frac{ax}{2}} = (c_1 + c_2 x)e^{-\frac{ax}{2}} \qquad (8.12)$$

where c_1, c_2, A and B are arbitrary constants to be determined by given conditions

Example 8.1 (p.246): Solve the following differential equation:

$$\frac{d^2u(x)}{dx^2} + 5\frac{du(x)}{dx} + 6u(x) = 0 \qquad \text{(a)}$$

Solution:

We have a = 5 and b = 6, by comparing Equation (a) with the typical differential equation in Equation (8.1) will lead to:

$a^2 - 4b = 5^2 - 4x6 = 25 - 24 = 1 > 0$ - a Case 1 situation with $\sqrt{a^2 - 4b} = \sqrt{1} = 1$

Consequently, we may use the standard solution in Equation (8.6) for the general solution of Equation (a):

$$u(x) = e^{-\frac{ax}{2}}\left(c_1 e^{\sqrt{a^2 - 4b}\, x/2} + c_2 e^{-\sqrt{a^2 - 4b}\, x/2}\right)$$

or

$$u(x) = e^{-5x/2}\left(c_1 e^{x/2} + c_2 e^{-x/2}\right) = c_1 e^{-2x} + c_2 e^{-3x}$$

where c_1 and c_2 are arbitrary constants to be determined by given conditions

Example 8.2 (p.246): Solve the following differential equation with given conditions:

$$\frac{d^2u(x)}{dx^2} + 6\frac{du(x)}{dx} + 9u(x) = 0 \qquad \text{(a)}$$

with given conditions: u(0) = 2 (b)

and $$\left.\frac{du(x)}{dx}\right|_{x=0} = 0 \qquad \text{(c)}$$

Solution:

Again by comparing Equation (a) with the typical differential equation in Equation (8.1), we have: a = 6 and b = 9.. Further examining $a^2 - 4b = 6^2 - 4x9 = 36 - 36 = 0$, leading to special Case 3 in Equation (8.12) for the solution:

$$u(x) = c_1 e^{-\frac{ax}{2}} + c_2 x e^{-\frac{ax}{2}} = (c_1 + c_2 x)e^{-\frac{ax}{2}} \qquad (8.12)$$

or

$$u(x) = (c_1 + c_2 x)e^{-\frac{6}{2}x} = (c_1 + c_2 x)e^{-3x} \qquad \text{(d)}$$

Using Equation (b) for Equation (d) will yield $c_1 = 2$, resulting in $u(x) = (2 + c_2 x)e^{-3x}$ (e)

Differentiating Equation (e) with condition in Equation (c) will lead to the following result:

$$\left.\frac{du(x)}{dx}\right|_{x=0} = \left[e^{-3x}(c_2) - 3e^{-3x}(2 + c_2 x)\right]_{x=0} = c_2 - 6 = 0$$

We may thus solve for $c_2 = 6$

Hence the complete solution of Equation (a) is: $u(x) = 2(1 + 3x)e^{-3x}$

8.3 Application of 2nd- Order Homogeneous Differential Equations for Free Mechanical Vibration Analysis (p.246)

8.3.1 What is mechanical vibration and resulting consequences?

Mechanical vibration is a form of ***oscillatory* motion** of a solid, a structure, a machine, or a vehicle induced by mechanical means.

The amount of movement in these solids and structures is called "***amplitude***". The amplitudes of vibrating solids vary with time. Such variations may be either regularly or in random fashions.

Oscillatory motion of solids with their amplitudes vary with fixed time interval called "***period***", and the reciprocal of the period is the "***frequency***" of the vibratory motions.

Consequences of mechanical vibrations:

It can be immediate, such as in the case of resonant vibration with rapid increase of magnitudes of vibration, resulting in immediate and unexpected **catastrophically structural failures**, or it can induce damages accumulated by long-term vibrations with low amplitudes. The latter form of vibrations may result in the failure of the machine or structure due to **fatigue** of the materials that make the machines or structures.

8.3.2 Common Sources for Mechanical Vibrations:

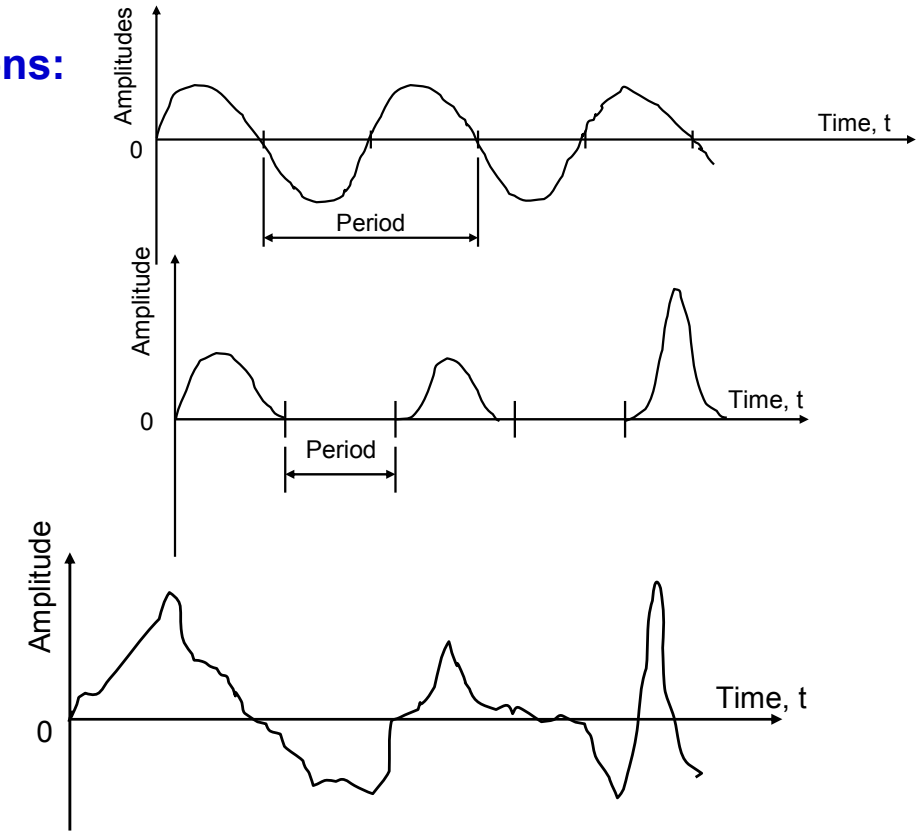
(1) Application of time-varying mechanical forces or pressure.
(2) Fluid induced vibrations due to intermittent forces of wind, tidal waves, etc.
(3) Application of pressures associated with acoustics and ultrasonic waves.
(4) Random movements of supports, for example, seismic forces.
(5) Application of thermal, magnetic forces, etc.

8.3.3 Common types of Mechanical Vibrations:

(1) With constant amplitudes and frequencies:

(2) With variable amplitudes but constant frequencies:

(3) With random amplitudes and frequencies:



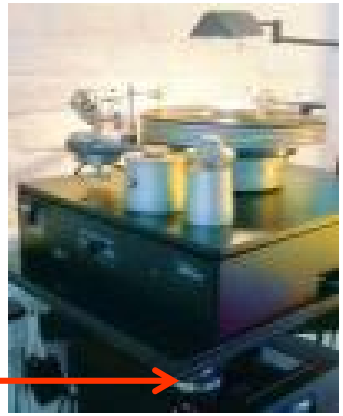
Mitigation of Mechanical Vibrations in mechanical systems:

Mechanical vibrations, in the design of mechanical systems, are normally undesirable occurrences, and engineers would normally attempt to either reduce it to the minimum appearance, or eliminate it completely.

"Vibration Isolators" are commonly designed and used to minimize vibration of mechanical systems, such as shown in the following cases:

Benches for high-precision instruments

Vibration isolators



Suspension of heavy-duty truck

Vibration isolators

Design of vibration isolators requires analyses to quantify the amplitudes and frequencies of the vibratory motion of the mechanical system – a process called "**mechanical vibration analysis**"

8.3.4 **The three types of mechanical vibration analyses by mechanical engineers** (p.247):

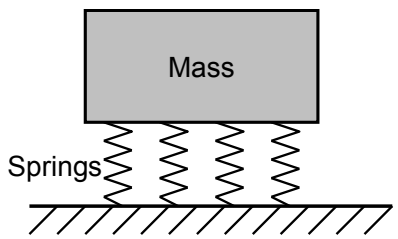
Mechanical vibration requires: Mass, spring force (elasticity), damping factor and initiator

8.3.4.1 Free vibration analysis:

The mechanical system (or a machine) is set to vibrate from its initial equilibrium condition by an instantaneous disturbance (either in the form of a force or a displacement).
This disturbance does not exist after the mass is set to vibrate.

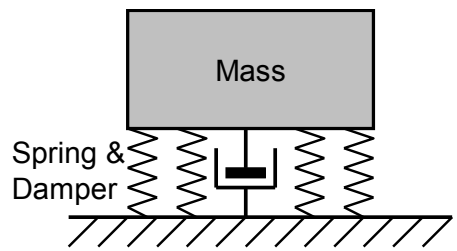
There are two types of free vibrations:

- Simple mass-spring system:



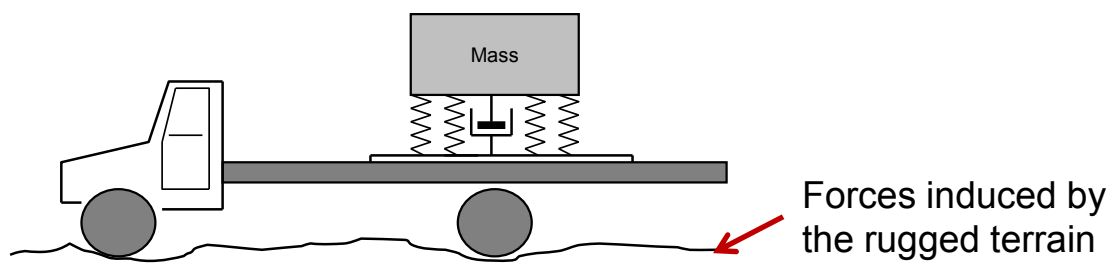
and

- Damped vibration system:



8.3.4.3 Forced vibration analysis (p.248):

Vibration of the mechanical system is induced by cyclic loading <u>at all times</u>.



Modal analysis

To identify natural frequencies of a solid machine at various possible modes of vibration

8.4 **Physical Modeling of Mechanical Vibrations: Simple mass-spring system** (p. 249)

Minimum requirement for Mechanical vibration: a MASS attached to an ELASTIC SUPPORT

The simplest model for mechanical vibration analysis is a **MASS-SPRING** system as illustrated in Figure 8.5:

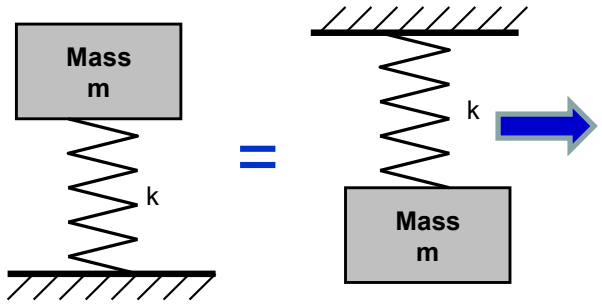
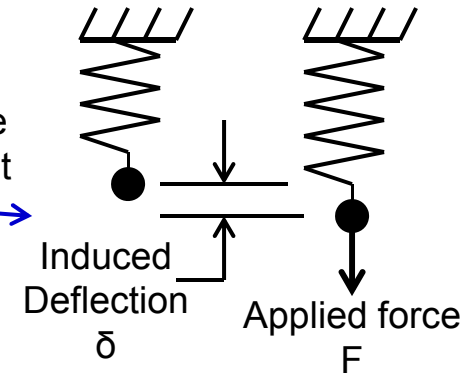


Figure 8.5 Simple mass-spring system

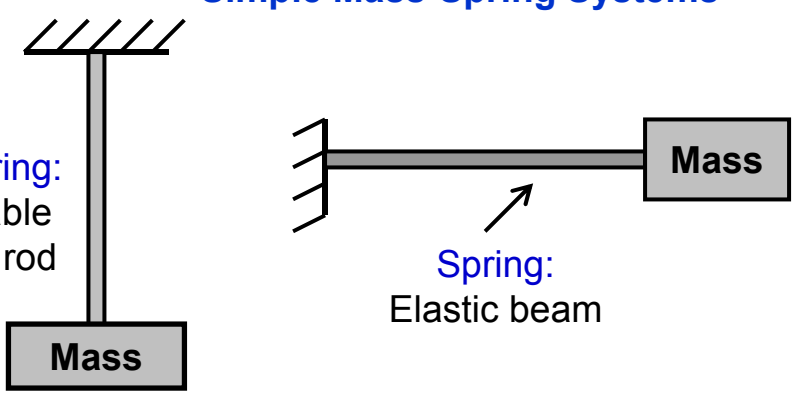
with m = mass, and
k = spring constant

- k is defined as the amount of force required to deflect a certain amount of the spring = F/δ
- So, k has a unit of lb_f/in or N/m
- k is a property of a given spring

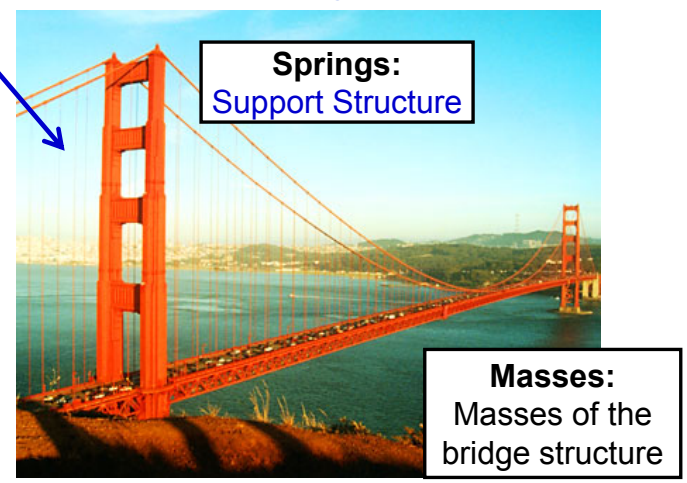


- The spring in this system is to support the mass
- Springs in the system need **not** to be "coil" springs
- Any ELASTIC solid support can be viewed as a "spring"

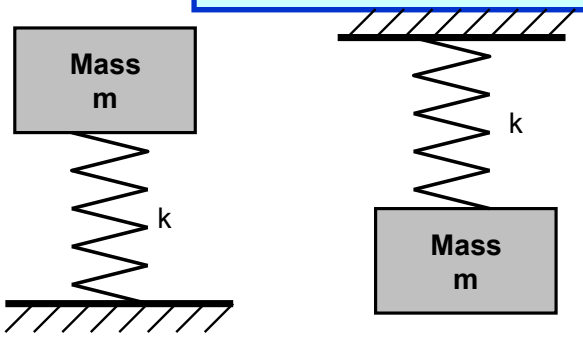
Simple Mass-Spring Systems



Complex System



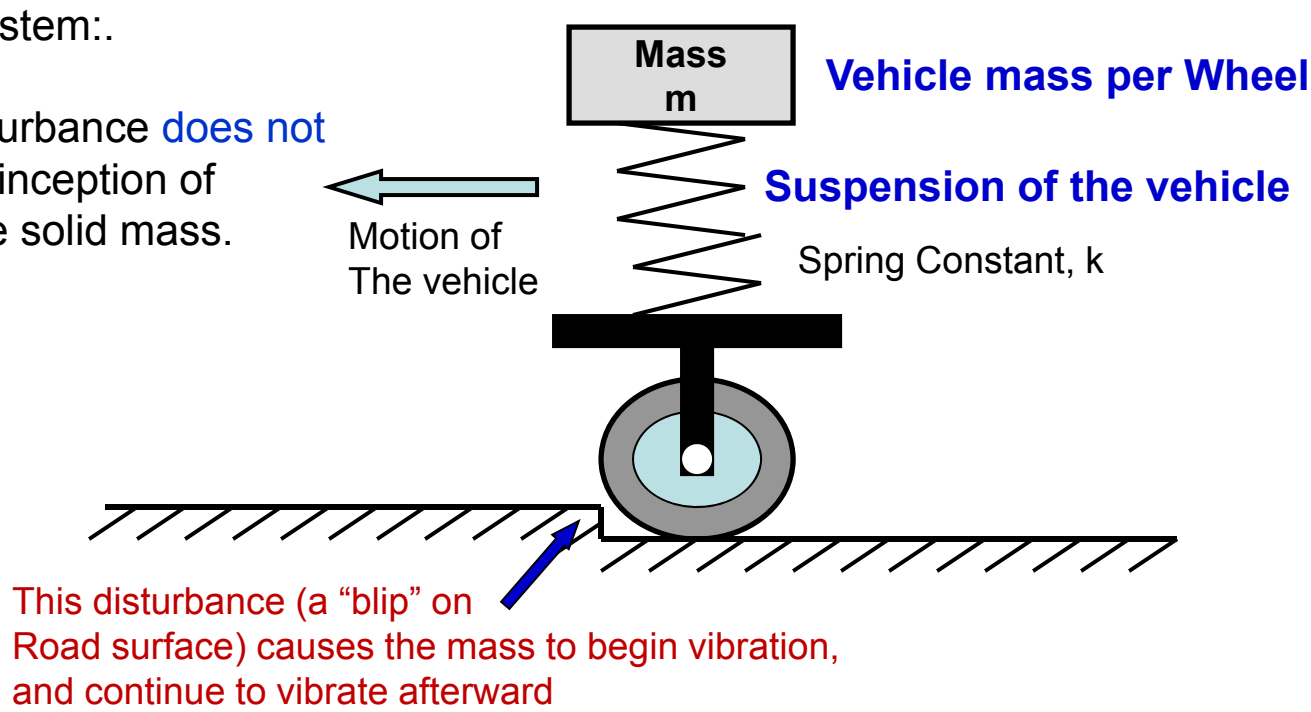
A Case of Simple Mass-Spring Systems in Free Vibration



It takes a **MASS** and **SPRING** (or elastic) support to get the mass to vibrate

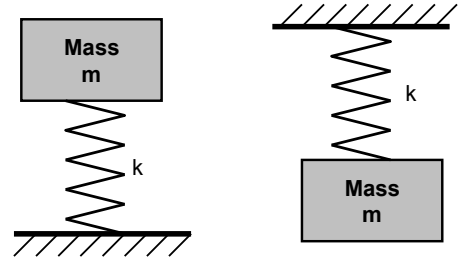
The physical phenomena of solids in free vibration is that the vibration of the solid is induced by an instantaneous disturbance either in the form of a force or deformation of the supporting spring, such as illustrated in the vibration of a vehicle induced by its suspension system:.

This initial disturbance does not exist after the inception of vibration of the solid mass.

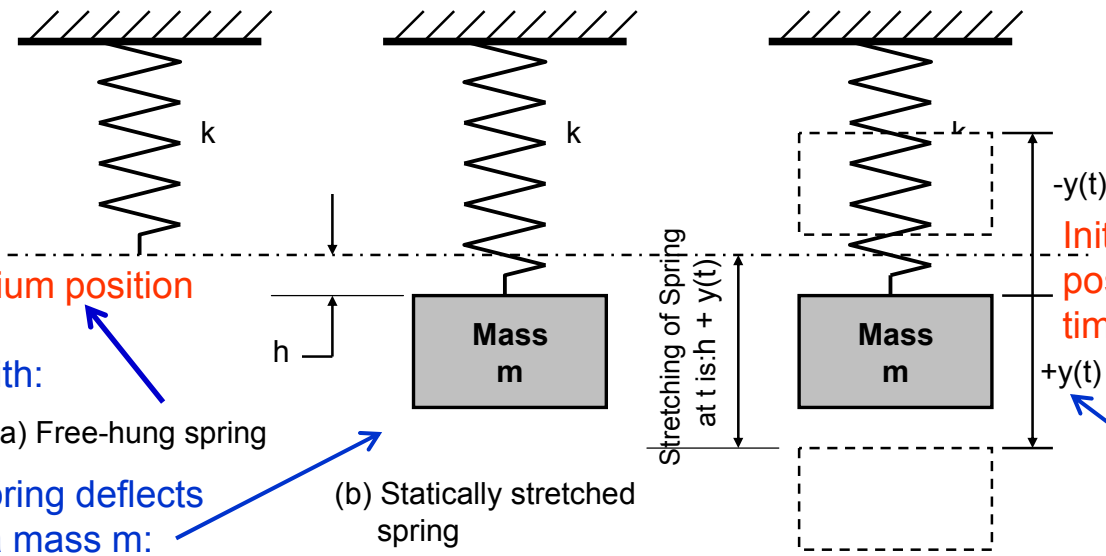


8.4 **Mathematical Formulation of Simple Mass-Spring Systems in Free Vibration** (p.249)

As we mentioned in previous occasions: "The simplest physical model for mechanical vibration analysis involves a **MASS** and a **SPRING** (or an elastic support) as illustrated in Figure 8.5



Mathematical Formulation for Free Vibration of a mass:



Initial equilibrium positionwith a mass at time zero

(1) We will Begins with:

(a) Free-hung spring

(2) The free-hung spring deflects upon attaching a mass m:

(b) Statically stretched spring

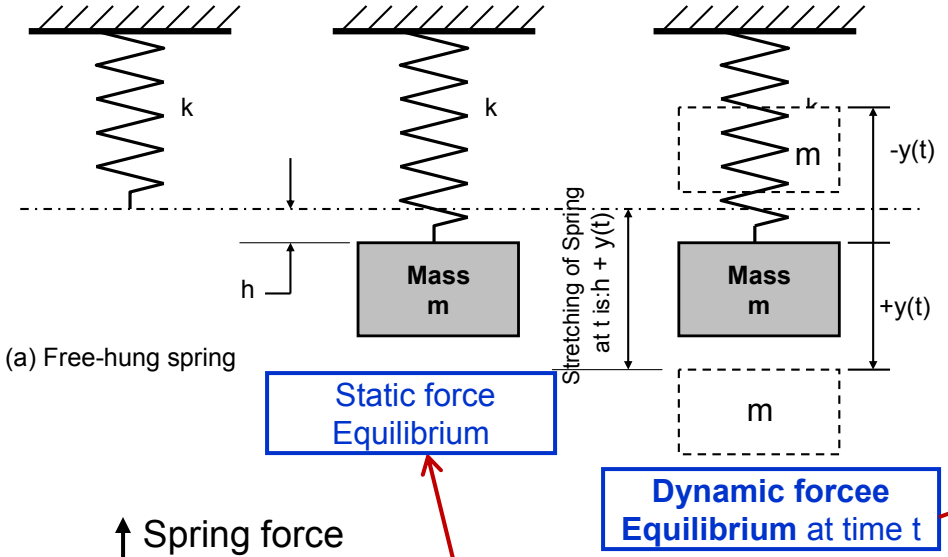
(c) A vibrating mass at time t

- Sign convention: +ve downward

(3) A small *instantaneous* "push-down" is applied to the mass and release quickly..
We can expect the mass to bounce up & down passing its initial equilibrium position. Due to simultaneous effects of the recoil of the spring and the dynamic forces associated with the motion of the mass in variable velocities.

Mathematical formulation of a vibrating mass in a Mass-Spring System

Based on an assumption with **no** air resistance against the motion off the mass and spring. Formulation is on the equilibrium of of both static and dynamic forces.

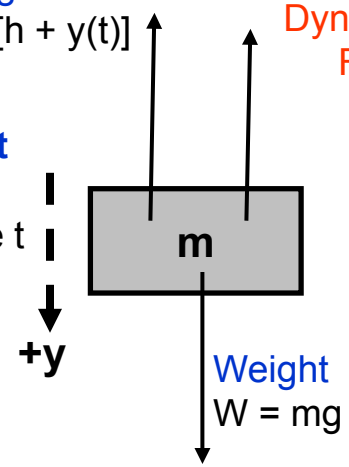


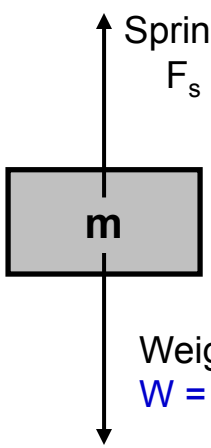
Forces: Weight (**W**); Spring force (**F_s**)
Dynamic force, **F(t)**:

Spring force:
$F_s = k[h + y(t)]$

Dynamic (Inertia) Force, F(t)

Displacement +y(t):
Position at time t





$$+\downarrow \sum F_y = W - F_s = 0$$

mg=kh

Equilibrium of forces acting on the mass at given time t satisfies the Newton's 1st Law:

$$+\downarrow \sum \left[-F(t) - F_s + W\right] = 0$$

But since we have the dynamic force to be: $F(t) = m\dfrac{d^2y(t)}{dt^2}$

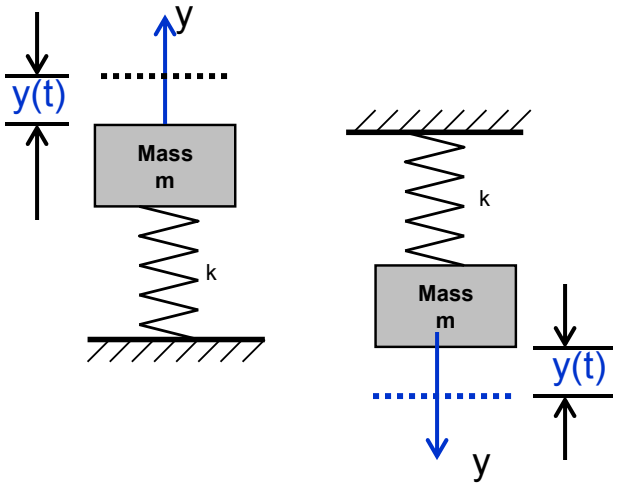
and the spring force to be $F_s = k[h + y(t)]$, we should have:

$$-m\frac{d^2y(t)}{dt^2} - k\left[h + y(t)\right] + mg = 0$$

But mg = kh from the static equilibrium condition,and after substituting it into the above equation, we have the following 2nd order differential equation for the instantaneous position y(t) for the vibrating mass:

$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = 0 \qquad (8.14)$$

Solution of differential equation (8.14) for simple mass-spring vibration



$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = 0 \qquad (8.14)$$

where y(t) = instantaneous position of the mass

Re-writing the above equation in the following form:

$$\frac{d^2y(t)}{dt^2} + \frac{k}{m}y(t) = 0 \qquad (8.14a)$$

The solution of Equation (8.14) can be obtained by comparing Equation (8.14a) with the typical 2nd order differential equation in Equation (8.1):

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \qquad (8.1)$$

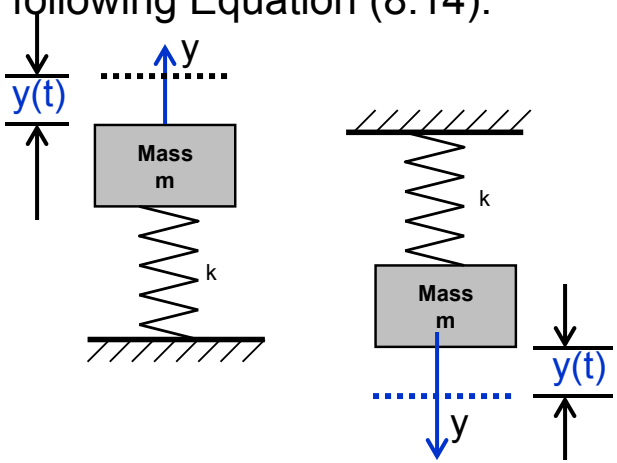
We will find that a = 0 and b = k/m after comparing Eqs..(8.14a) and (8.1). The solution of Equation (8.1) depends on the discriminator: $a^2 - 4b$. Since k = spring constant which is a property of the spring and m = mass of the vibrating solid, the equivalent coefficient b is a +ve real number. Consequently, we will have: $a^2 - 4b = 0 - 4(k/m) < 0$, which is a Case 2 for the solution, as shown in Equation(8.8), or

$$y(t) = A\,Cos\sqrt{\frac{k}{m}}\,t + B\,Sin\sqrt{\frac{k}{m}}\,t \qquad (8.15)$$

where A, B are arbitrary constants to be determined by given conditions

Physical senses of solution of differential equation (8.14) for simple mass-spring vibration:

The differential equation for the instantaneous position of the solid mass, y(t) satisfies the following Equation (8.14):



$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = 0 \qquad (8.14)$$

where y(t) = instantaneous position of the mass

Upon re-writing the equation in the form:

$$\frac{d^2y(t)}{dt^2} + \frac{k}{m}y(t) = 0 \qquad (8.14a)$$

The general solution of Equation (8.14) is:

$$y(t) = c_1\,Cos\,\omega_o\,t + c_2\,Sin\,\omega_o\,t \qquad (8.16)$$

where c_1 and c_2 are arbitrary constants to be determined by given conditions, and

$$\omega_o = \sqrt{\frac{k}{m}} \qquad (8.16a)$$

is called the "*circular*", or "*angular frequency*" of the mass-spring vibration system. Often, it represents the "***natural frequency***" of the simple mass-spring system. The unit is rad/s.

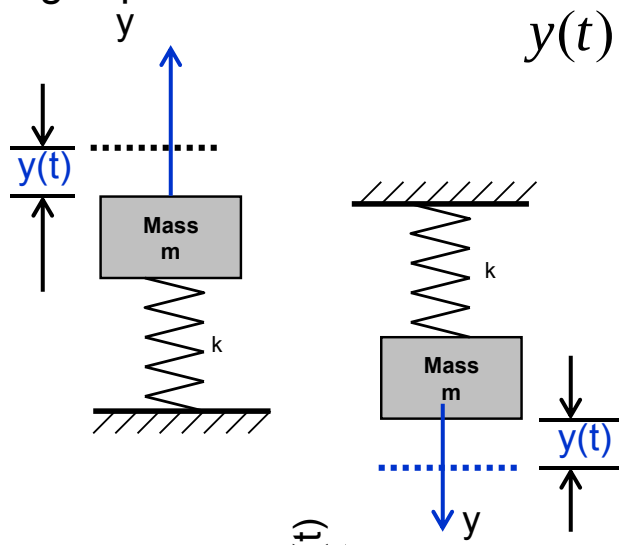
Corresponding to the angular frequency ω_o is the real frequency of the vibration in the following expression::

$$f = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (8.17)$$

Graphical representation of free-vibration of simple mass-spring systems (p.252):

The mathematical solution for the instantaneous position of the vibrating mass can be obtained by the following expression:

$$y(t) = c_1 \, Cos\, \omega_o \, t + c_2 \, Sin\, \omega_o \, t \qquad (8.16)$$



The solution y(t) consists of cosine and sine functions of variable t (the time)

So, it is an **oscillatory** function, oscillating about the "zero-time" axis, with the **amplitudes of vibration y(t)** illustrated in Figure 8.8:

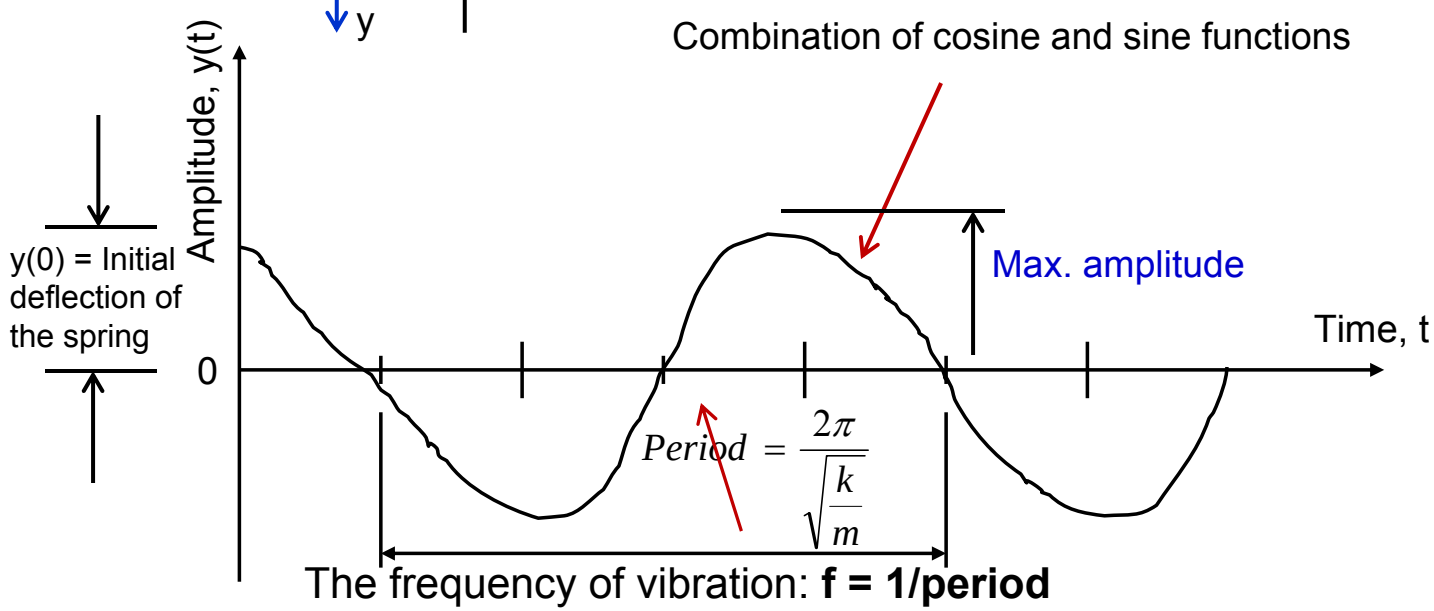


Figure 8.8 The oscillatory motion of the solid mass in a simple mass-spring system

Example 8.3 (p.253): An unexpected case for engineers to consider in their design and operation of an unloading process of a heavy machine.

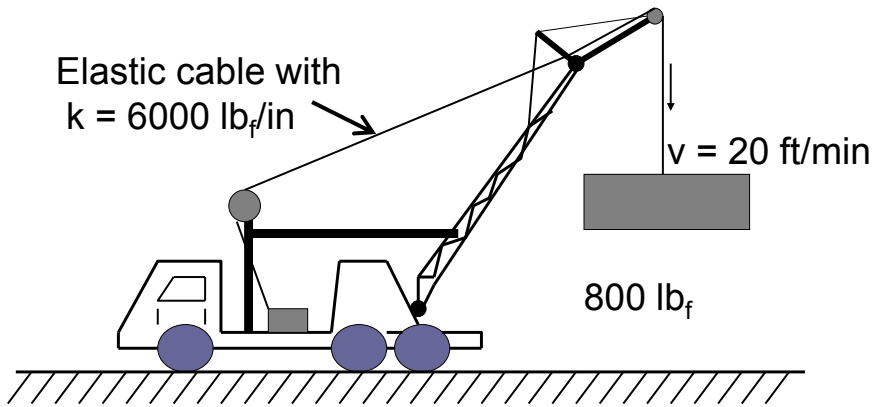


Figure 8.10

Description of the problem:
A truck is unloading a heavy machine weighing 800 lb$_f$ by a crane as illustrated in Figure 8.10. The cable used to lift the freight was suddenly seized (jammed) at time t from a descending velocity of v = 20 ft/min

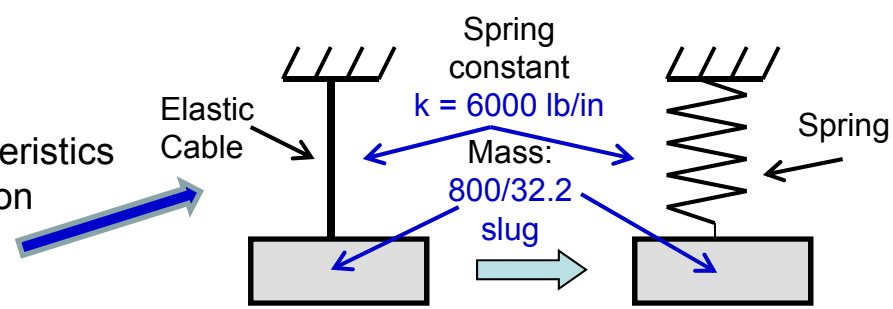
One may expect the heavy machine undergoing An "up-down-up" vibration after such seizure.

Determine the following:

(A) The frequency of vibration of the machine that is seized from its descending
(B) The maximum tension in the cable induced by the vibrating machine,
(C) The maximum stress in the cable if the stranded steel cable is 0.5 inch in diameter
(D) Would the cable break if its ultimate tensile strength (UTS) of the cable material is 40,000 psi?

Solution:

Because the machine is attached to an elastic cable, which has the same characteristics as a "spring," we may simulate this situation to a simple mass-spring systems as:



The frequency and amplitudes of the vibrating machine can thus be evaluated by the expressions derived for the simple mass-spring system.

Example 8.3 – Cont'd

(a) **The frequency of vibration** of the machine is given in Equation (8.17)..

The circular frequency is:

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{6000\,x12}{800/32.2}} = 53.83\ rad/s$$

which leads to the frequency to be:

$$f = \frac{\omega_o}{2\pi} = 8.57\ \ cycles/s$$

(b) The **maximum tension in the cable**:

The maximum tension in the cable is determined with its maximum total elongation during the vibratory motion of the machine after the cable is seized.
To get the amplitude of the vibrating machine, we need to solve a differential equation that has the form as shown in Equation (8.14) satisfying the appropriate conditions.
The following formulation is obtained:

The differential equation: $$m\frac{d^2y(t)}{dt^2} + k\,y(t) = 0 \qquad (8.18)$$

With given conditions: y(0) = 0 and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 20\ ft/\min = 0.3333\ ft/s$ (a)

Initial velocity (velocity at the time of seizure)

The solution of Equation (8.14) is: $y(t) = c_1 \, Cos\, \omega_o t + c_2 \, Sin\, \omega_o t$ (8.16)

or $$y(t) = c_1 \, Cos\, 53.83\, t + c_2 \, Sin\, 53.83 t \qquad \text{(b)}$$

with ω_o = 53.83 rad/s as computed in Part (a) of the solution.

The arbitrary constants c_1 and c_2 in Equation (b) can be determined by using the given conditions in Equation (a), with c_1 = 0 and c_2 = 0.0062 in Equation (b).

We thus have the amplitude of the vibrating machine in the following form:

$$y(t) = 0.0062 \, Sin\, 53.83 t \qquad \text{(c)}$$

from which, we obtain the maximum amplitude from Equation (c) to be:

y_{max} = 0.0062 ft

The corresponding maximum tension in the cable is:

$$T_m = k\, y_{\max} + W = (6000\, x 12)\, x\, 0.0062 + 800 = 1246 \; lb_f$$

where W is the weight of the machine

(C) The equivalent **maximum stress in the cable** is obtained by the following expression: with A being the cross-sectional area of the steel cable:

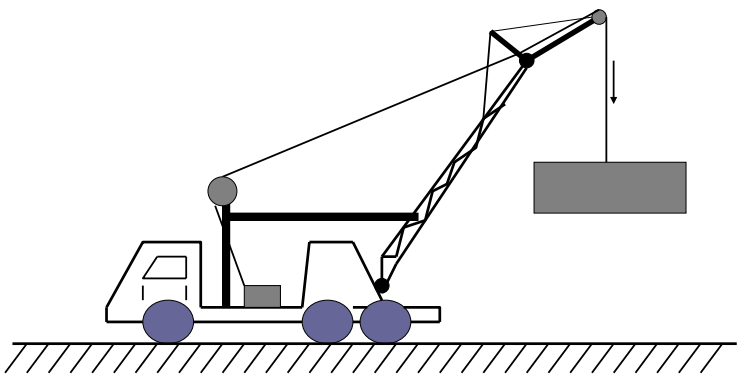
$$\sigma_{\max} = \frac{T_m}{A} = \frac{1246}{\frac{\pi (0.5)^2}{4}} = 6346 \;\; psi$$

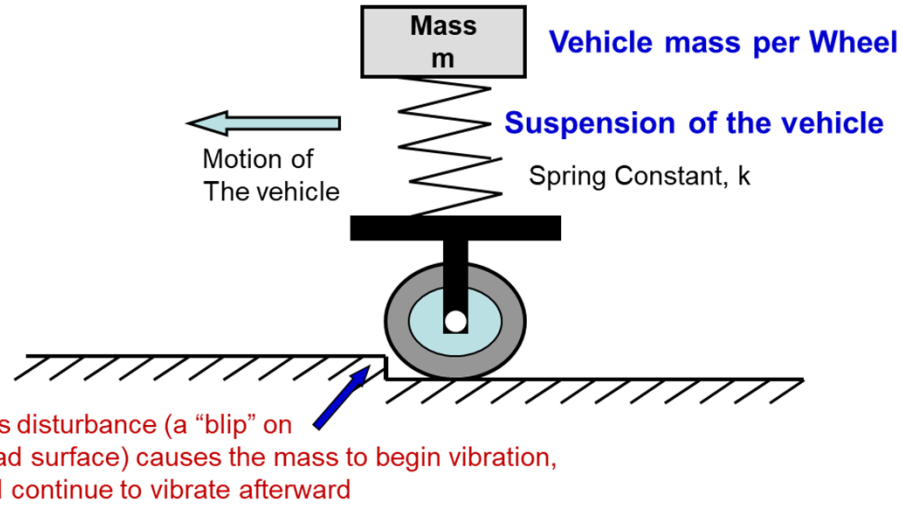
(d) Interpretation of the analytical result:

The cable will not break because the maximum induced stress σ_{max} = 6346 psi << uts where uts is the ultimate tensile strength of the cable material = 40,000 psi.

On the Application of Simple Mass-Spring Systems in Free Vibration Analysis:

We have demonstrated that simple mass-spring systems are used to derive comprehensive math
models for free vibration of solids. It has demonstrated the principle of free vibration by two practical problems described in Example 8.3 and the vibration of a vehicle chassis when it is hit by a small blip on the road on which it cruises. Both these applications are illustrated below:





Deficiency of simple mass-spring systems:

$$y(t) = c_1 \, Cos\, \omega_o t + c_2 \, Sin\, \omega_o t$$

The mass oscillates FOREVER!!

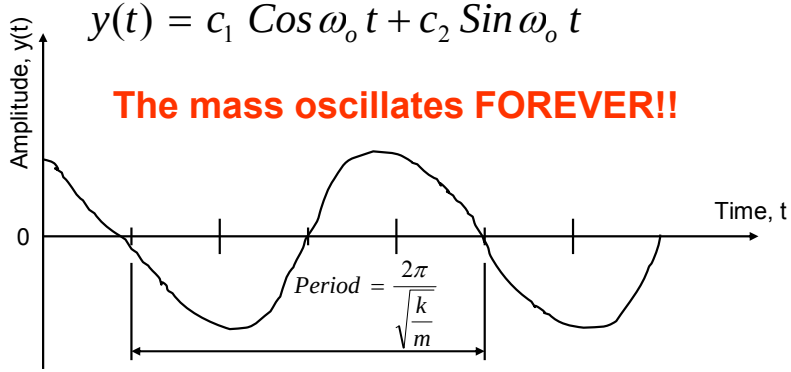


Figure 8.8 Graphic representation of analytical results of a simple mass-spring system

Despite the simplicity of the math model as shown in Equation (8.14), and its general solution in Equation (8.15), the analytical results as illustrated in Figure 8.8 bears little reality with the amount of oscillatory the amplitudes of the vibrating solids never decay-a phenomenon that cannot be realistic.

- Proper modifications to the simple mass-spring model must be made to make it more **realistic**.

8.5 Simple Damped Mass-Spring Systems in Free Vibration (p.254)

Question: What makes free-vibration of a mass-spring system to stop in reality??

Answer: It is the "**damping effect**" that makes the free vibration of mass-spring system to stop after soke time t.

So, "Damped" free vibration of solids is a more realistic phenomenon

Conceivable Sources for Damping in Mechanical Vibrations:

- Resistance by the air surrounding the vibrating mass – hard to model analytically
- Internal friction of the spring during deformations – a material science topic, hard to model

8.5.1 A practical physical model of Damped Mass-Spring Systems in free vibration analysis:

Because damping of a simple mass-spring vibration system is induced by the resistance of the surrounding air to the moving mass, we can use an "air cylinder" with adjustable air vent to regulate the air resistance to the moving mass such illustrated in Figure 8.11 (a):

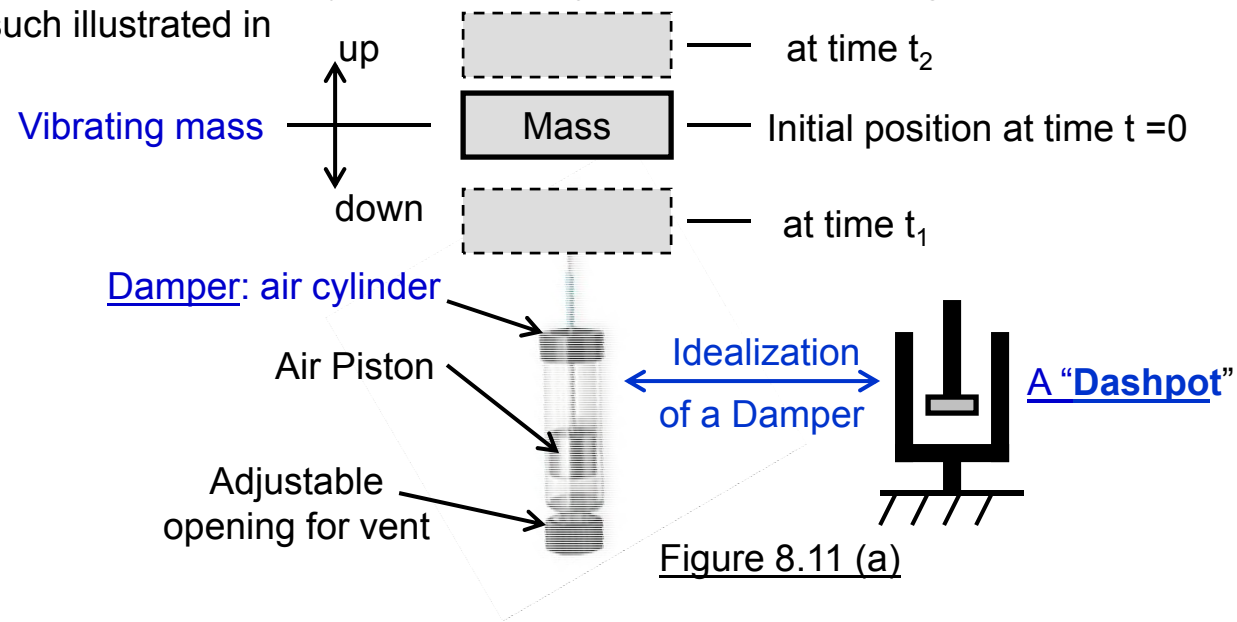
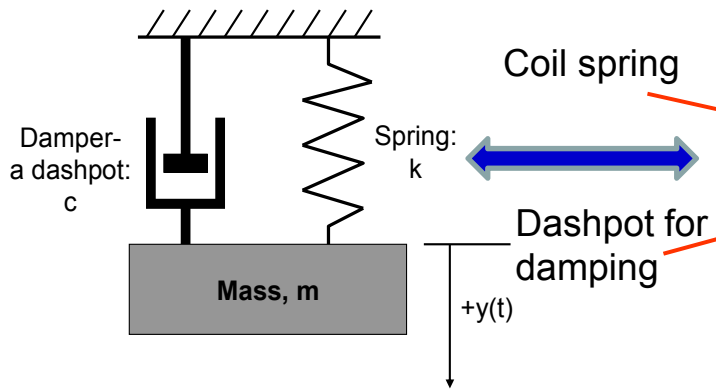


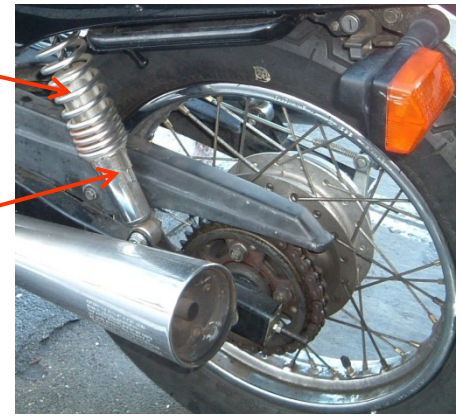
Figure 8.11 (a)

Physical modeling of Damped Mass-Spring Systems in Free Vibration Analysis (p.255)

Real-world application with "**Coilovers**"** in vehicle suspension

** A "coilover" consists of a dashpot surrounded by a coil spring





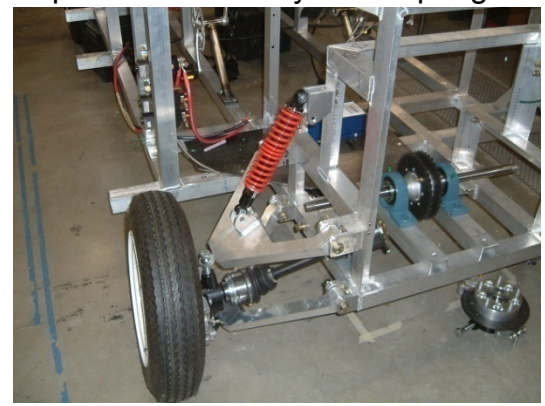


Figure 8.11(b) Physical model of damped vibration

The damper in the physical model is characterized by a damping coefficient c – similar to the situation of a spring characterized by spring constant k.

The damping coefficient c is specified by manufacturer of the damper (or a dashpot)

Because the corresponding damping force is related to the air resistance to the movement of the mass, and the air resistance R is proportional to the velocity of the moving mass. Mathematically, we have:

$$R(t) \propto Velocity\ of\ moving\ mass \left(= \frac{dy(t)}{dt} \right)$$

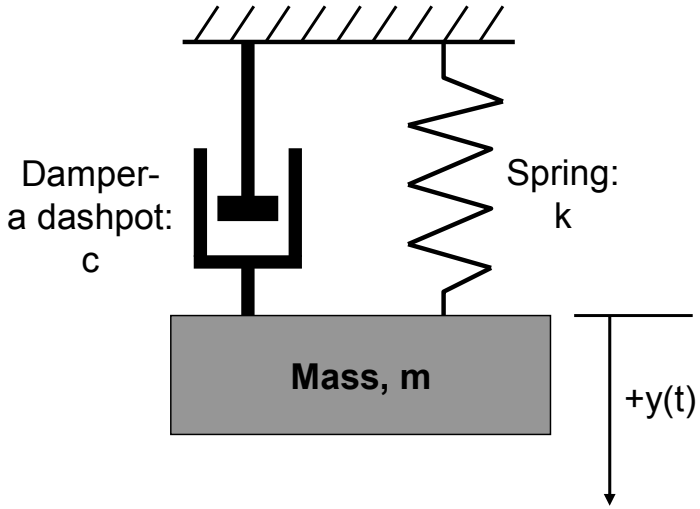
where y(t) relates to the distance the mass travels to and from its initial equilibrium position,

Consequently, the damping force R(t) has the form:

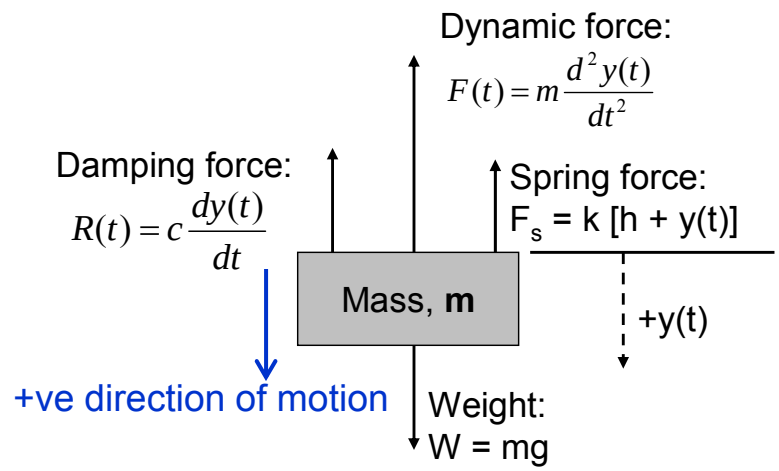
$$R(t) = c\frac{dy(t)}{dt} \qquad (8.19)$$

in which c = damping coefficient, normally supplied by the vendor of the damper.

8.5.2 Mathematical modeling of Damped Mass-Spring Systems in Free Vibration Analysis



The mathematical expression of this physical model can be obtained by following similar procedure for free vibration of simple mass-spring system, but with the inclusion of the additional damping force as:



By Newton's 1st Law in dynamic equilibrium at time t:

$$+\downarrow \sum F_y = -F(t) - R(t) - F_s + W = 0$$

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + k\,y(t) + kh - mg = 0$$

or

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + k\,y(t) = 0 \qquad (8.20)$$

Equation (8.20) is a 2nd order homogeneous differential equation for the instantaneous position of the vibrating mass, y(t)

Solution of Eq. (8.20) for Damped Mass-Spring Systems in Free Vibration Analysis

$$m\frac{d^2y(t)}{dt^2} + c\frac{dy(t)}{dt} + k\,y(t) = 0 \qquad (8.20)$$

If we re-write the above equation in a different form:

$$\frac{d^2y(t)}{dt^2} + \frac{c}{m}\frac{dy(t)}{dt} + \frac{k}{m}y(t) = 0 \qquad (8.20a)$$

Now, if we compare Equation (8.20a) and the typical 2nd order homogeneous differential equation in Equation (8.1):

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = 0 \qquad (8.1)$$

we will have equivalences: $a = c/m$ and $b = k/m$ after comparing terms in Equations (8.20a) and (8.1)

We may obtain the solutions of Equation (8.20) depends on the signs of the discriminators ($a^2 - 4b$) or $(c/m)^2 - 4(k/m) > 0$ for Case1, or $=0$ for Case 2, or <0 ifor Case 3. Effectively, we will look for all the 3 possible cases in the following cases:

Case 1: $(c/m)^2 - 4(k/m) > 0$, or $c^2 - 4mk > 0$

Case 2: $(c/m)^2 - 4(k/m) = 0$ or $c^2 - 4mk = 0$

Case 3: $(c/m)^2 - 4(k/m) < 0$ or $c^2 - 4mk < 0$

8.5.3 Solution of Eq. (8.20) for Damped Mass-Spring Systems in Free Vibration Analysis:

Case 1: $c^2 - 4mk > 0$ (Over-damping situation):

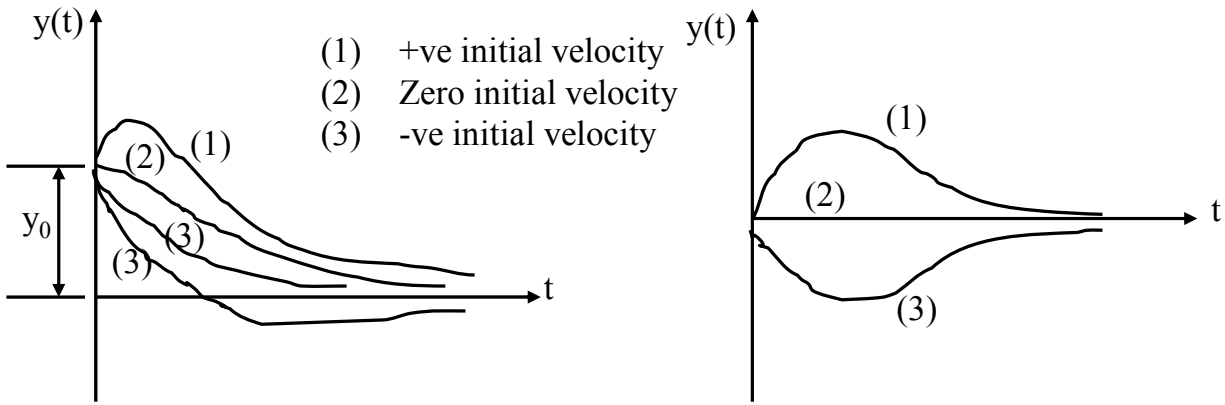
The solution in Equation (8.6) is applied:

$$y(t) = e^{-(c/2m)t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right) \qquad (8.23)$$

where A and B are arbitrary constants to be determined by two given conditions, and

$\Omega = \sqrt{c^2 - 4mk}\,/\,(2m)$

Graphical representation of the instantaneous position of the vibrating mass are:



(a) With +ve initial displacement, y_0 (b) With negligible initial displacement

Observations:

- There is **no** oscillatory motion of the mass,.
- There can be an initial increase in the displacement of the mass, followed by continuous decays in the amplitudes in the vibration, and
- The amplitudes of vibration usually decays quickly in time.

- It is a desirable situation in abating (or mitigating) mechanical vibration

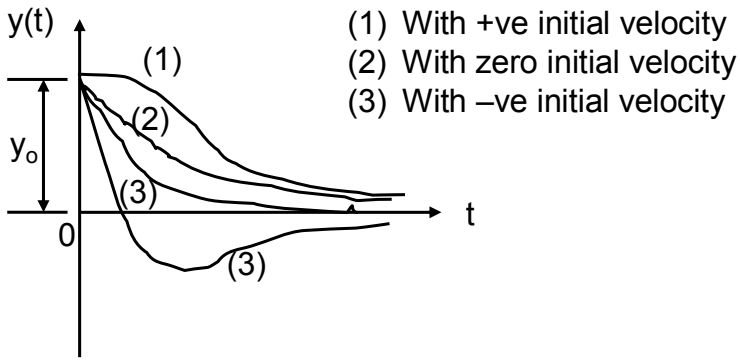
8.5.3 Solution of Eq. (8.20) for Damped Mass-Spring Systems in Free Vibration Analysis:-Cont'd

Case 2: $c^2 - 4mk = 0$ (Critical damping):

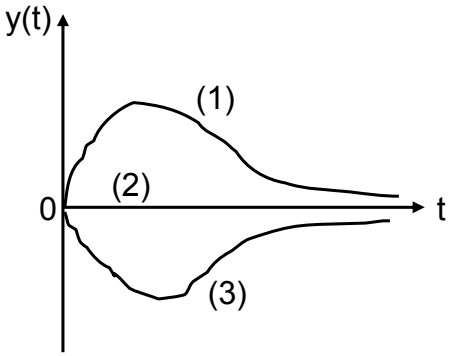
Solution of Equation (8.20) is in the form of Equation (8.12):

$$y(t) = e^{-\left(\frac{c}{2m}\right)t}\left(A + Bt\right) \tag{8.24}$$

Graphical representation of Equation (8.24) is:



(a) With +ve initial displacement



(b) With negligible initial displacement

Observations:

- There is no oscillatory motion of the mass by theory,
- Amplitudes reduce with time, but take longer to "die down" than in the case of "over-damping," and
- May become an unstable situation of vibration.

8.5.3 Solution of Eq. (8.20) for Damped Mass-Spring Systems in Free Vibration Analysis:-Cont'd

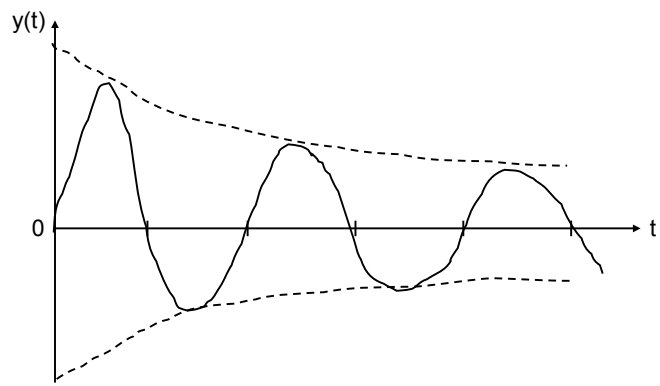
Case 3: $c^2 - 4mk < 0$ ((Under damping):

Solution of Equation (8.20) in this case is expressed in Equation (8.8)

$$y(t) = e^{-\left(\frac{c}{2m}\right)t}\left(A\,Cos\,\Omega t + B\,Sin\,\Omega t\right) \qquad (8.25)$$

where $\Omega = \sqrt{4mk - c^2}\,/\,(2m)$ and A, B are arbitrary constants

Graphical representation of Equation (8.25) is:



Observations:

- The only case of damped vibration that has oscillatory motion of the mass,
- The amplitudes of each oscillatory motion of the mass reduces continuously but they take **a long time to "die down,"** and
- "Under damping" is thus the **least** desirable situation in machine design.

8.6 Solution Method of Nonhomogeneous Second-Order Ordinary Differential Equations (p.258)

We have learned from Chapter 7 on the efforts required in solving homogeneous and nonhomogeneous first order differential equations, and also on the complexity of the solutions for nonhomogeneous differential equations.

Unlike what we learned in that chapter, there is no fix rule or particular solution method to follow in solving nonhomogeneous second order differential equations. What we will learn from this Chapter is a general guideline for the solution of this type of differential equations.

Non-homogeneous second order differential equations have broad applications in engineering analysis; however, we will focus on its applications in the following three areas in:

(1) Applications in forced vibration analysis.
(2) Application in “resonant vibration Analysis” of solid machine structures that often results in devastating structural failures, and
(3) Application in “near resonant vibration analysis.” This type of vibration usually takes long time for structural failure. But structural failure is mainly caused by the “fatigue” of the materials.

8.6.1 **Typical differential equation and solution:**

$$\frac{d^2u(x)}{dx^2} + a\frac{du(x)}{dx} + bu(x) = g(x) \qquad (8.26)$$

The nonhomogeneous term

Solution of Equation (8.26) consists of **TWO** components:

Solution u(x) = Complementary solution $u_h(x)$ + Particular solution $u_p(x)$

or

$$u(x) = u_h(x) + u_p(x) \qquad (8.27)$$

The complementary solution $u_h(x)$ is the solution of the homogeneous part of Equation (8.26), i.e.:

$$\frac{d^2u_h(x)}{dx^2} + a\frac{du_h(x)}{dx} + bu_h(x) = 0 \qquad (8.28)$$

Equation (8.28) is similar to the typical 2nd order homogeneous differential equation in Equation (8.1).Solutions are available in Equation (8.6) for Case 1 with a^2 -4b>0; Equation (8.7) for Case 2 with a^2-4b<0; and Equation (8.12) for Case 3 with a^2-4b = 0

8.6.3 Determination of particular solution $u_p(x)$ (p.259)

There is **NO** fixed rule for deriving $u_p(x)$. However, the following guideline may be used to determine $u_p(x)$ by ASSUMING a function that is **SIMILAR** to the nonhomogeneous part of the differential equation, e.g., g(x) in Equation (8.26):

Table 8.1 Guidelines for choosing assumed forms of up(x)

g(x) Given in Eq. (8.26) with specified coefficients (a,b,c and d)	**$u_p(x)$** Assumed solution with unknown coefficients, (A_o, A_1, A_2 A_3 and A_4, or A,B, C,D) need to be determined
Polynomial functions of order n: $g(x) = ax^4 + b\,x^2 + cx + d$ (order 4)	Polynomial functions of order n: $u_p(x) = A_o + A_1x + A_2x^2 + A_3x^3 + A_4x^4$ (order 4)
Trigonometric functions: g(x) = a sine or cosine function, or $g(x) = \tan(\alpha x)$,or $\cot(\alpha x)$,or $g(x) = \sec(\alpha x)$, or $\csc(\alpha x)$	with **Conjugate** trigonometric functions: $u_p(x) = A\cos(\alpha x) + B\sin(\alpha x)$ $u_p(x) = A\tan(\alpha x) + B\cot(\alpha x)$,or $u_p(x) = A\sec(\alpha x) + B\csc(\alpha x)$
Exponential functions: $g(x) = ae^{bx}$	Exponential functions: $u_p(x) = Ae^{bx}$
Combination of functions: $g(x) = ax^3 + b\cos(\alpha x) + ce^{-dx}$	Combination of similar functions: $u_p(x) = (Ax^3+Bx^2+Cx+D)$ $+ [E\cos(\alpha x) + F\sin(\alpha x)] + Ge^{-dx}$

The unknown coefficients in the assumed $u_p(x)$ are determined by comparing terms after their substituting into Equation (8.26), as shown in the next slide:

The coefficients in assumed $u_p(x)$ are determined by comparing terms after its substitution into the entire differential equation in Equation (8.26):

$$\frac{d^2u_p(x)}{dx^2} + a\frac{du_p(x)}{dx} + bu_p(x) = g(x) \qquad (8.29)$$

Your assumed $u_p(x)$

Self studies on Examples (8.4), (8.5) and (8.7). We will use Example 8.6 to demonstrate the solution of the nonhomogeneous differential equation using the proposed method, and A special form of $u_p(x)$ to solve special case of nonhomogeneous differential equations.

Example 8.6 (p.262): Solve the following 2nd order nonhomogeneous differential equation::

$$\frac{d^2y(x)}{dx^2} - \frac{dy(x)}{dx} - 2y(x) = Sin\,2x \qquad (a)$$

Solution:

Equation (a) is a nonhomogeneous equation. So its solution takes the following forms as in Equation (8.27):

$$y(x) = y_h(x) + y_p(x) \qquad (b)$$

The complementary solution $y_h(x)$ in Equation (b) is obtained from the homogeneous part of Equation (a) as:

$$\frac{d^2y_h(x)}{dx^2} - \frac{dy_h(x)}{dx} - 2y_h(x) = 0 \qquad (c)$$

The solution of Equation (c) is by Case 1 with a^2-4b>0, or:

$$y_h(x) = c_1\,e^{-x} + c_2\,e^{2x} \qquad (d)$$

To determine the particular solution $y_p(x)$ following the guidelines in Table 8.1:

The nonhomogeneous part in Equation (a) with g(x) = sin 2x. Consequently, we will assumed $y_p(x)$ to include **BOTH** trigonometric functions sine and cosine functions:

$$y_p(x) = A \text{ Sin } 2x + B \text{ Cos } 2x \qquad \text{(e)}$$

which leads to:

$$\frac{dy_p(x)}{dt} = 2ACos2x - 2BSin2x \quad \text{and} \quad \frac{d^2y_p(x)}{dt^2} = -4ASin2x - 4BCos2x$$

Substituting $y_p(x)$ in Equation (e) and its derivatives into Equation (a):

$$\frac{d^2y_p(x)}{dx^2} - \frac{dy_p(x)}{dx} - 2y_p(x) = Sin2x \Rightarrow$$

(- 4A Sin 2x – 4B Cos 2x) –
(2A Cos 2x – 2BSin 2x) –
-2(A Sin 2x + B Cos 2x) = Sin 2x

g(x)

After re-arranging terms, we get:

(- 6A + 2B) Sin 2x + (-6B – 2A) Cos 2x = Sin 2x. from the above equality

By comparing the coefficients of the terms on both sides of the above expression, we get:

6A = 2B = 1 and – 2A – 6B = 0, from which we solve for: A = - 3/20 and B = 1/20

The particular solution is thus: $y_p(x)$ = -3 Sin2x/20 + Cos 2x/20, which leads to the solution of the Differential equation in Equation (a) to be:

$$y(x) = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 e^{2x} + \left(-\frac{3}{20}Sin2x + \frac{1}{20}Cos2x\right)$$

where c_1 and c_2 are arbitrary constants to be determined by the given specified conditions

8.6.4 Special Case in Determining Particular Solution $u_p(x)$ (p.263)

These cases involve at least one term in the complementary solution of the DE,i.e., $u_h(x)$ coincides with the term of the function in the nonhomogeneous part of the DE, i.e. g(x).

Example of the special case – **Example 8.8**:

The differential equation:

$$\frac{d^2u(x)}{dx^2} + 4u(x) = 2\,Sin\,2x \qquad \text{(a)}$$

Following the usual procedure, we will get the complementary solution first by solving:

$$\frac{d^2u_h(x)}{dx^2} + 4u_h(x) = 0 \qquad \text{(b)}$$

with a solution: $u_h(x) = c_1 \text{ Cos } 2x + c_2 \text{ Sin } 2x$ (c)

where c_1 and c_2 are arbitrary constants

We realize the 2nd term in the solution of $u_h(x)$ in Equation (c) is of the similiar form of g(x) = 2 sin2x in Equation (a). So, it is a special case. We will see from the following derivation of $u_p(x)$ by the "normal" way will lead us to NOWHERE as we will see form the following derivation!

Since the nonhomogeneous part of Equation (a) is g(x) = 2 sin2x – a trigonometric function, the "normal" way would having us assuming the particular solution $u_p(x)$ in the form:

$$u_p(x) = A \text{ Cos } 2x + B \text{ Sin } 2x \qquad \text{(d)}$$

Substituting the $u_p(x)$ in Equation (d) into Equation (a) will lead to the following ambiguous equality:

$$(0)Cos\,2x + (0)Sin2x = 2\,Sin2x$$

There is no way can we solve for the coefficients A and B in Equation (d). **Another way of obtaining $u_p(x)$ is needed**

8.6.4 Special Case in Determining Particular Solution $u_p(x)$ – Cont'd

Particular solution for special cases:

Let us modify the assumed $u_p(x)$ in Equation (d) by adding "x" in the assumed $u_p(x)$ for the special case:

$$u_p(x) = x\,(A \text{ Cos } 2x + B \text{ Sin } 2x) \qquad \text{(e)}$$

Now if we follow the usual procedure with the modified $u_p(x)$ in Equation (e) to Equation (a), we need first to derive the following derivatives as:

$$\frac{du_p(x)}{dx} = A(-2xSin2x + Cos2x) + B(2xCos2x + Sin2x) \qquad \text{(f)}$$

and

$$\frac{d^2u_p(x)}{dx^2} = A[-4xCos2x - 2Sin2x - 2Sin2x]$$
$$= B[-4xSin2x + 2Cos2x + 2Cos2x] \qquad \text{(g)}$$

Upon substituting the above modified $u_p(x)$ in Equation (e) and the derivatives in Equations (f) and (g) into Equation (a), we will have:

(-4Ax Cos 2x – 2A Sin 2x – 2A Sin 2x – 4Bx Sin2x + 2B Cos 2x + 2B Cos 2x)
+ (4Ax Cos 2x + 4Bx Sin 2x) = 2 Sin 2x

from which we get: A = -1/2 and B = 0, which lead to: $u_p(x) = -\frac{x}{2}Cos2x$ (h)

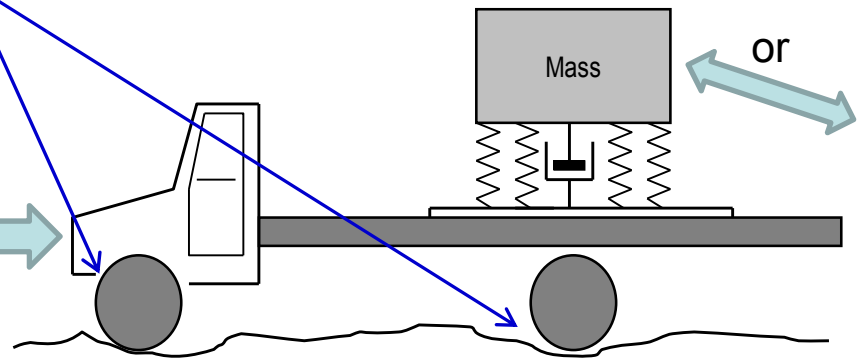
The complete general solution of Equation (a) is thus possible by a summation of $u_h(x)$ in Equation (c) and the $u_p(x)$ in Equation (h) to give:

$$u(x) = u_h(x) + u_p(x) = c_1\,Cos\,2x + c_2\,Sin\,2x - \frac{x}{2}Cos\,2x$$

8.7 Application of 2nd Order Nonhomogeneous Differential Equations to Forced Vibration Analysis

We have learned from Section 8.3.4.3 on the definition of forced vibration of machine structures. There are generally two types of forces involved in this kind of forced vibrations as illustrated below. We will focus on the latter type of forces: the cyclic forces with frequencies designated by ω rad/sec because this is the kind of forces that would induce catastrophic structural failures.

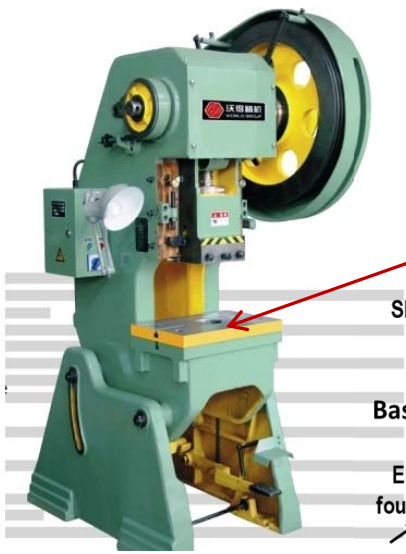
(A) By <u>irregular forces from rough road surface</u>:

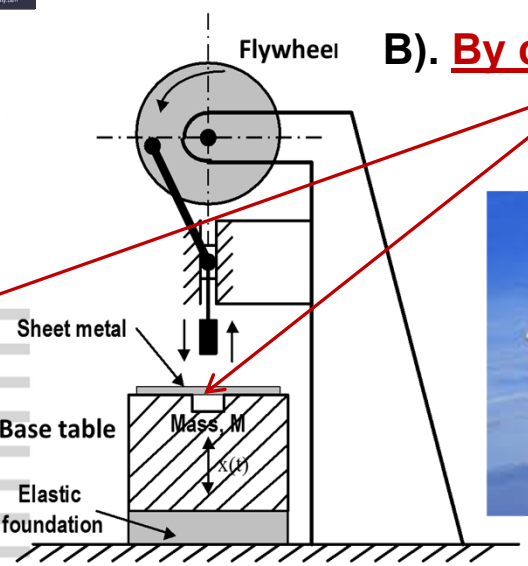


or

Vehicle on Suspension systems

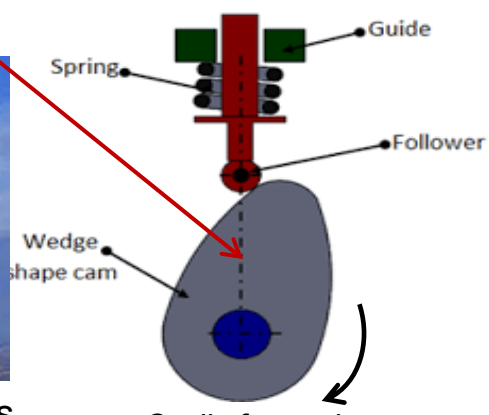
B). <u>By cyclic forces:</u>





Cyclic forces in a stamping machine

Cyclic aerodynamic forces by propellers

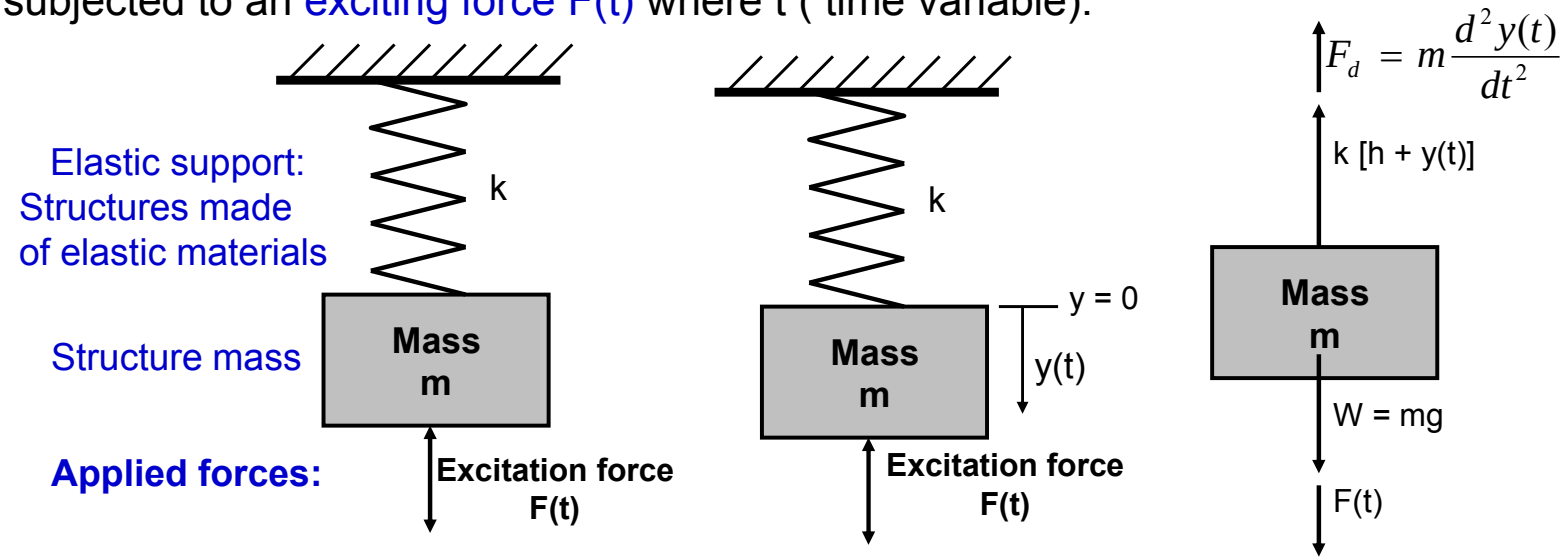


Cyclic forces by cam-spring forces

8.7 Forced Vibration Analysis

8.7.1 Derivation of differential equation (p.264):

The simplest physical model for forced vibration is a simple mass-spring system subjected to an exciting force F(t) where t (time variable):



The mathematical model for the above physical arrangement can be derived by using Newton's First law:

$$+\uparrow \sum F_y = 0 \rightarrow -F_d - k[h + y(t)] + W + F(t) = 0$$

with $F_d = m\dfrac{d^2y(t)}{dt^2}$ from Newton's 2nd law

The differential equation for the instantaneous amplitudes of the vibrating mass under the influence of force F(t) becomes:

$$m\frac{d^2y(t)}{dt^2} + ky(t) = F(t) \qquad (8.31)$$

Equation (8.31) is a nonhomogeneous 2nd order differential equation

8.7 Forced Vibration Analysis – Cont'd

8.7.1 Derivation of differential equation:

Forced Vibration of a Mass-Spring System subject to Cyclic Forces with frequency ω:

If we assume the applied force F(t) in Equation (8.31) is of cyclic nature following a cosine function, such as:

$$F(t) = F_o \text{ Cos } \omega t \qquad (8.32)$$

where F_o = maximum magnitude of the force, and ω is the circular frequency of the applied cyclic force that can be graphically displayed in Figure 8.17:

:

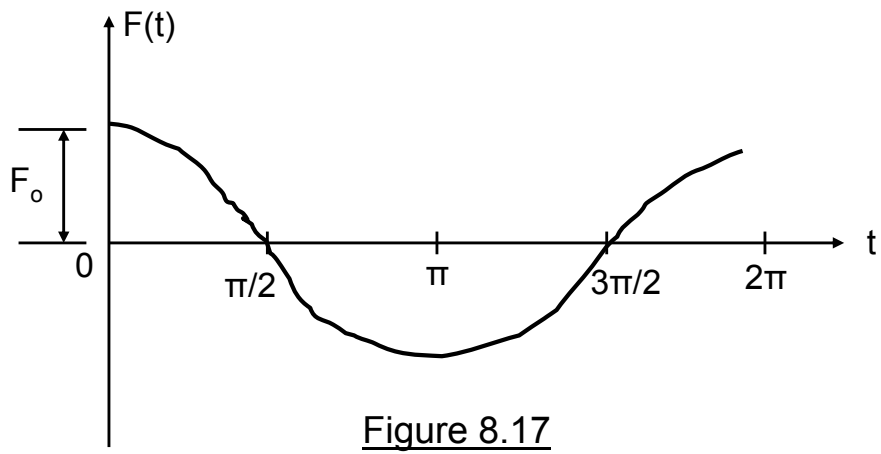


Figure 8.17

Upon substituting the expression of F(t) in Equation (8.32) into Equation (8.31), we have the governing differential equation for the amplitudes of the vibrating mass as:

$$m\frac{d^2 y(t)}{dt^2} + k\, y(t) = F_0 \, Cos\, \omega t \qquad (8.33)$$

8.7 Forced Vibration Analysis – Cont'd

8.7.1 Derivation of differential equation:

Solution of Equation (8.33): $$m\frac{d^2y(t)}{dt^2} + k\,y(t) = F_0\,Cos\,\omega t$$

or in a different form: $$\frac{d^2y(t)}{dt^2} + \frac{k}{m}y(t) = \frac{F_o}{m}Cos\,\omega t \qquad (8.33a)$$

or in yet another form: $$\frac{d^2y(t)}{dt^2} + \omega_o^2\,y(t) = \frac{F_o}{m}Cos\,\omega t \qquad (8.33b)$$

in which $\omega_0 = \sqrt{\frac{k}{m}}$ is the circular frequency of the mass-spring system in Section 8.4.1 (NOTE: ω_0 is an <u>inherent **property**</u> of the mass-spring "structure")

Equation (8.33b) is a non-homogeneous 2nd order differential equation, and its solution is:

$$y(t) = y_h(t) + y_p(t)$$

The complementary solution $y_h(t)$ is obtained from the homogeneous part of Equation (8.33b):

$$\frac{d^2y_h(t)}{dt^2} + \omega_0^2\,y_h(t) = 0 \qquad (8.33c)$$

The general solution of Equation (8.33c) is:

$$y_h(t) = c_1\,Cos\,\omega_o t + c_2\,Sin\,\omega_o t \qquad (4.33d)$$

where c_1 and c_2 are arbitrary constants to be determined by given conditions of the problem.

The form of particular solution of Equation (8.33b) can be obtained by following the guideline in Table 8.1 as:

$$y_p(t) = A \text{ Cos } \omega t + B \text{ Sin } \omega t \qquad (8.34)$$

We will have: $\dfrac{dy_p(t)}{dt} = -A\omega \, Sin\omega t + B\omega \, Cos\,\omega t$ and $\dfrac{d^2y(t)}{dt^2} = -A\omega^2 \, Cos\,\omega t - B\omega^2 \, Sin\,\omega t$

Upon substituting the above into Equation (8.33b) with $y(t) = y_p(t)$:

$$\frac{d^2 y_p(t)}{dt^2} + \omega_0^2 \, y_p(t) = \frac{F_0}{m} Cos\,\omega t$$

We have: $\left(-A\omega^2 \, Cos\omega t - B\omega^2 \, Sin\omega t\right) + \omega_0^2 \left(ACos\omega t + BSin\omega t\right) = \dfrac{F_0}{m} Cos\omega t$

Upon comparing terms on both sides of the above equality, we will get the following relationshops:

$\left(-A\omega^2 + \omega_0^2 \, A\right) = \dfrac{F_0}{m}$ for the terms associated with Cosωt, leading to: $A = \dfrac{F_0}{m(-\omega^2 + \omega_0^2)}$

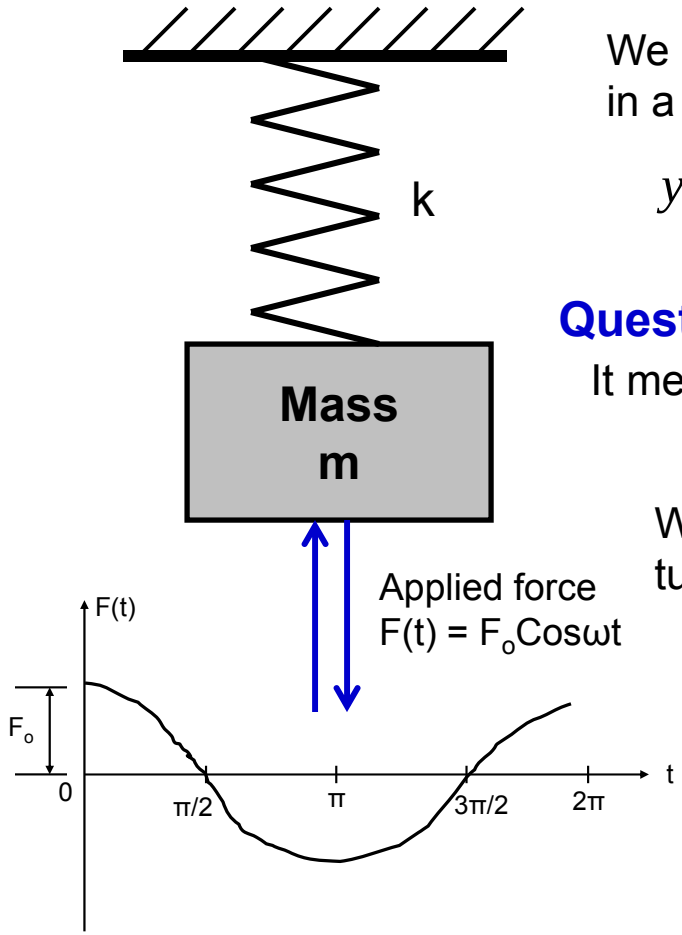
and for the term of Sinωt: $\left(-\omega^2 + \omega_0^2\right)B = 0$ leading to: B = 0

Thus, we have: $y_p(t) = \dfrac{F_0}{m(-\omega^2 + \omega_o^2)} Cos\omega t$

The complete solution of Equation (8.33b) for forced vibration by cyclic force $F(t) = F_o$ Cosωt is:

$$y(t) = c_1 \, Cos\omega_0 t + c_2 \, Sin\omega_0 t + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} Cos\,\omega t \qquad (8.35)$$

where c_1 and c_2 are the arbitrary constants determined by specified initial conditions



We realize the solution on the amplitudes of the vibrating mass in a forced vibration systems is:

$$y(t) = c_1 Cos\omega_0 t + c_2 Sin\omega_0 t + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} Cos\omega t \qquad (8.35)$$

Question: What will happen in the special case of: $\omega = \omega_0$?

It means that **the frequency of the applied cyclic force (ω) = the inherent natural frequency of the structure (ω_0)**

We will observe that the amplitude y(t) in Equation (8.35) turn into an unrealistic situation:

$y(t) \rightarrow \infty$ **Meaning the amplitude of vibration becomes infinity under any case at all times**

which is **not** physically possible!!!

So, the solution in Equation (8.35) is **NOT** realistic!!
And an **alternative solution** needs to be derived for the case of $\omega = \omega_0$

HENCE, for the case of **resonant vibration** in the situation with:

The frequency of the excitation (applied) force (**ω**)
= The circular frequency (NATURAL FREQUENCY) of the Mass- spring system (**ω_0**)

requires a special solution method as will be presented in the next slide:

8.7.2 **The Resonant Vibration Analysis** (p.266)

Because we have the situation with **$\omega = \omega_0$**, Equation (8.33) now can be expressed in the Following form:

$$m\frac{d^2y(t)}{dt^2} + k\,y(t) = F_0\,Cos\,\omega_0\,t \qquad \text{(a)}$$

We may express the complementary solution of Equation (a) as we did before:

$$y_h(t) = c_1\,Cos\,\omega_o t + c_2\,Sin\,\omega_o t$$

We notice that the same term of "$Cos\omega_0 t$" appears in the above solution as in the non-homogeneous part (on the right-hand side) of Equation (8.33b). Consequently, the particular solution of Equation (8.33b) for the case of $\omega=\omega_o$ fits the "special case" category for the solution $y_p(t)$..

Let us now assume the particular solution to be in a special case (in Section 8.6.4) to be:

$$y_p(t) = t\,(A\,Cos\,\omega_0 t + B\,Sin\,\omega_0 t)$$

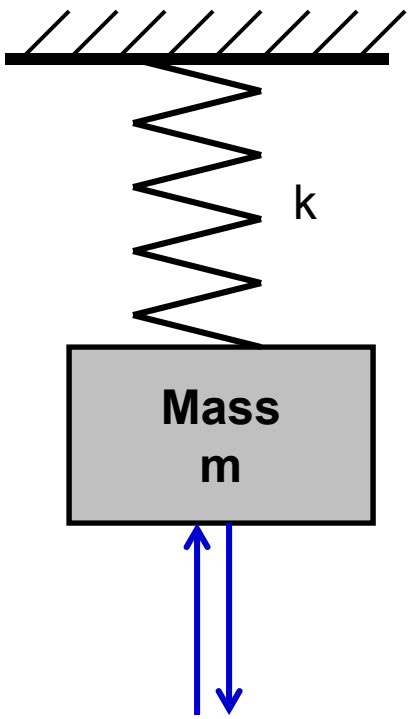
By following the same procedure as we used in solving non-homogeneous Equation (a) to get:

A = 0 and $B = \dfrac{F_0}{2m\omega_0}$

Hence

$$y_p(t) = \frac{F_0}{2m\omega_0}\,t\,Sin\,\omega_0\,t$$

8.7.2 The Resonant Vibration Analysis (p.267) – Cont'd



Applied force

$F(t) = F_o \cos\omega_0 t$

The amplitude of the vibrating mass in resonant vibration is:

$$y(t) = c_1 \, Cos\,\omega_0 t + c_2 \, Sin\,\omega_0 t + \frac{F_0}{2m\omega_0} \; t \; Sin\,\omega_0 t \qquad (8.36)$$

With the following graphical representation of the amplitude fluctuation of the vibrating mass y(t) shown in Figure 8.18:

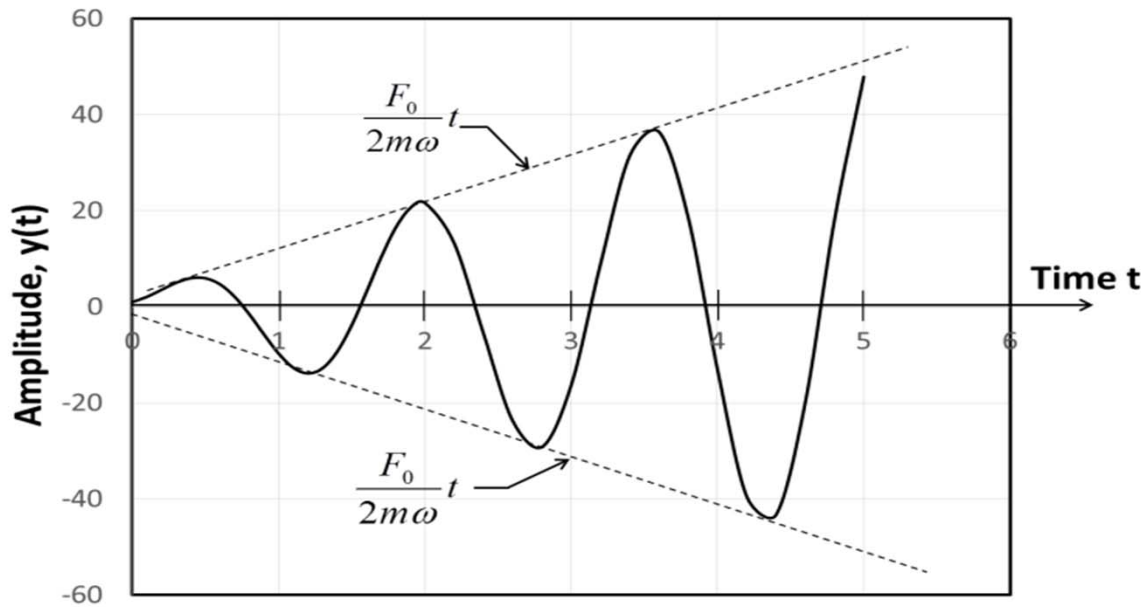


Figure 8.18 Amplitude of a resonant vibration of a solid mass

We notice the peculiar phenomenon from the graphic representation of the solution in Equation (8.36) for the case in which:

The frequency of the applied cyclic force (ω)

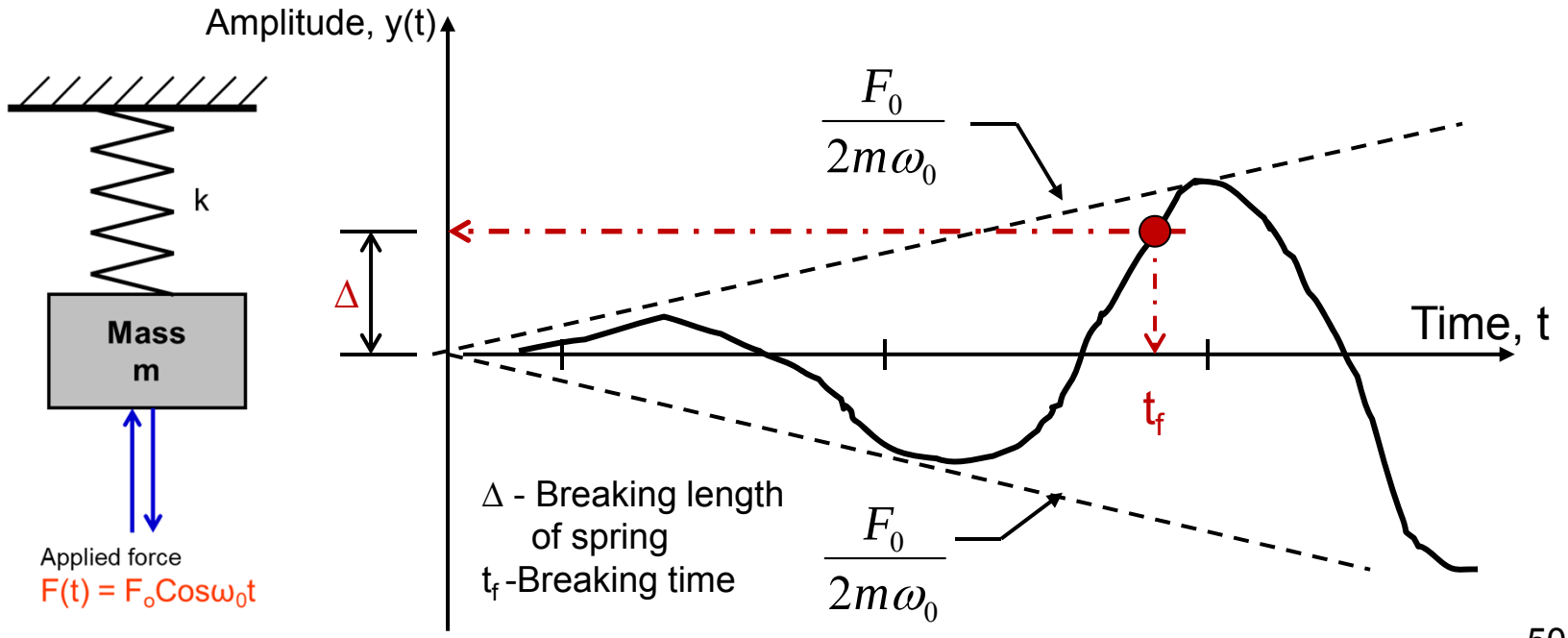
= the inherited frequency of the mass-spring system (ω_o)

The MAXIMUM amplitude y(t) increases CONTINUOUSLY with time t at a rate of ($F_o/2m\omega$) without limit! THIS IS THE CASE OF WHAT IS CALLED **RESONANT VIBRATION**" in engineering analysis.

8.7.2 **The Resonant Vibration Analysis –** Cont'd

Prediction of time to failure of a mass-spring system in Resonant vibration:

- The amplitude of vibration of the mass y(t) in a mass-spring structure increases CONTINUOUSLY with time t (and often in very steep rates) in a RESONANT VIBRATION.
- The physical consequence to the structure is the rapid "DEFORMATION" (or STRETCHING of the supporting spring in a mass-spring situation."
- The attached spring will soon be "stretched" to break when the maximum allowed elongation Δ is beyond the limit of the spring material at time t_f (as found in Material's handbook)
- The estimated time of structural failure in resonant vibration of mass-spring system t_f may be obtained as shown in the following graph with t_f= the time at which the amplitude of vibrating mass reaches the elongation of he spring $\Delta = y(t_f)$.

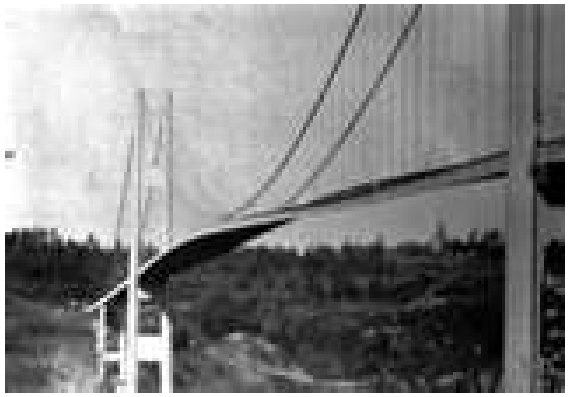


A Well-documented Catastrophic Structural Failure of Tacoma Narrow Bridge - A classical case of structure failure by Resonant Vibration

- The bridge was located in Tacoma, Washington, USA
- Started building on Nov 23, 1938
- Opened to traffic on July 1, 1940

- The bridge was 2800 feet long, 39 feet wide
- A 42 mph wind blew over the bridge in early morning of November 7, 1940

- The intermittent wind provided an external force with a periodic frequency that matched one of the natural structural frequency of the bridge structure that triggered a "resonant vibration of the bridge sturcture.
- The bridge began to gallop with increasing magnitudes



- Eventual structure failure at about 11 AM

No human life was lost. A small dog was perished because he was too scared to run for his life

Example 8.9 (p.268) Resonant vibration analysis of a "stamping" machine

A stamping machine applies hammering forces on metal sheets by a die attached to the plunger

The plunger moves vertically up-and-down by a flywheel spinning at a constant speed.

The constant rotational speed of the flywheel makes the impact force on the sheet metal, and therefore the supporting base, intermittent and cyclic

The heavy base on which the metal sheet is supported has a mass M = 2000 kg

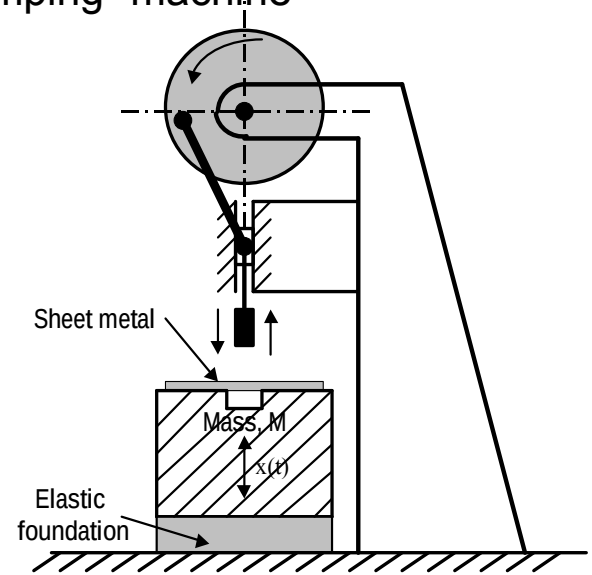


Figure 8.19

The force acting on the base can be described by a function: F(t) = 2000 Sin(10t), in which t = time in seconds

The base is supported by an elastic pad to absorb the cyclic impact forces with an equivalent spring constant $k = 2\times10^5$ N/m

Determine the following if the base is initially depressed down by an amount 0.005 m:

(a) The differential equation for the instantaneous position of the base, i.e., x(t)
(b) Examine if this is a resonant vibration situation with the applied load
(c) Solve for x(t)
(d) Should this be a resonant vibration, how long will take for the support to break at an elongation of 0.03 m?

Example 8.9 - Cont'd

Solution:

The situation can be physically modeled to be a mass-spring system:

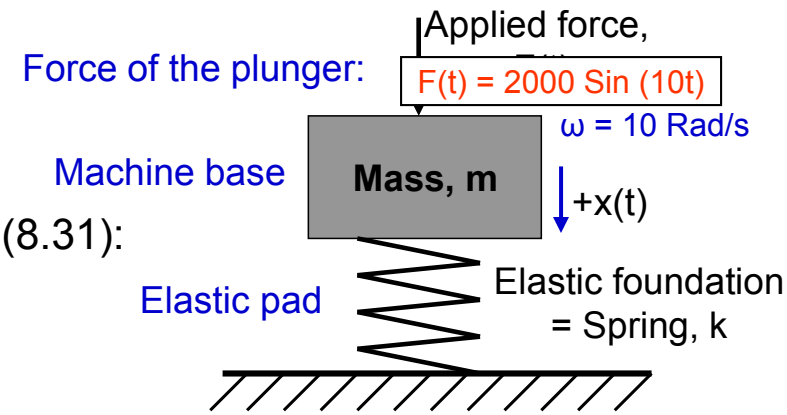


Figure 8.20

(a) The governing differential equation from Equation (8.31):

$$2000\frac{d^2x(t)}{dt} + 2x10^5\ x(t) = 2000\,Sin10t \qquad \text{(a)}$$

with initial conditions:

$$x(0) = 0.005\text{ m} \quad \text{and} \quad \left.\frac{dx(t)}{dt}\right|_{t=0} = 0 \qquad \text{(b)}$$

(b) To check if this is a resonant vibration situation:

Let us calculate the Natural (circular) frequency of the mass-spring system in Figure 8.20:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2x10^5}{2x10^3}} = 10\ Rad/s \qquad \text{(c)}$$

We notice the frequency of the excitation force, ω = 10 Rad/s, which is the same as the natural frequency of the structure ω_0, we thus conclude that it is a resonant vibration of the base structure because **$\omega_0 = \omega$.**

Example 8.9 - Cont'd

(c) Solution of differential equation in Equation (a):

It is a nonhomogeneous differential equation, so its solution x(t) consists of two parts:

$$x(t) = x_h(t) + x_p(t)$$

By now, we know how to solve for the complementary solution $x_h(t)$ in the following form:

$$x_h(t) = c_1 Cos10t + c_2 Sin10t \qquad \text{(d)}$$

Because it is a resonant vibration – a special case for solving non-homogeneous 2nd order differential equations, the particular solution $x_p(t)$ will take the form:

$$x_p(t) = t\,(ACos10t + BSin10t) \qquad \text{(e)}$$

By substituting the $x_p(t)$ in Equation (e) into the differential equation into Equation (a), and comparing terms on both sides, we will have the constants A and B in Equation (c) computed as: A = -1/20 and B = 0.

We will thus have the particular solution

$$x_p(t) = -(t/20)\cos 10t \qquad \text{(f)}$$

By substituting Equations (d) and (f) into the expression for x(t), we will have the general solution of Equation (a) to be:

$$x(t) = x_h(t) + x_p(t) = c_1 Cos10t + c_2 Sin10t - \frac{t}{20} Cos10t \qquad \text{(g)}$$

Example 8.9 - Cont'd

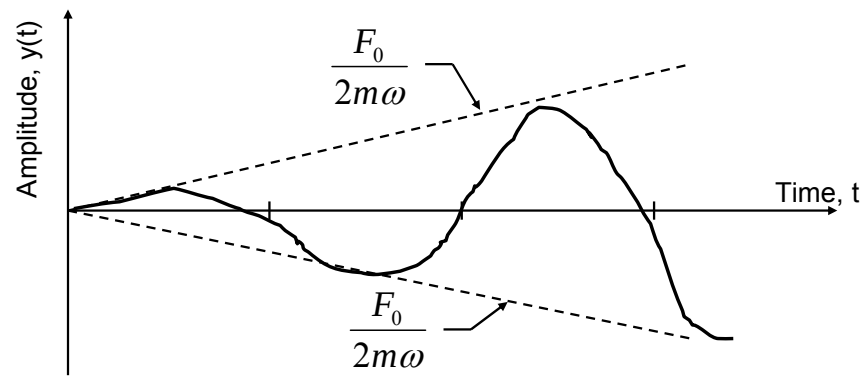
Apply the two specified initial conditions in Equation (b) into the above general solution will result in the values of the two arbitrary constants with: c_1 = 0.005 and c_2 = 1/200

The complete solution of Equation (a) is thus:

$$x(t) = 0.005Cos\ 10t + \frac{1}{200}Sin10t\ - \frac{t}{20}\text{Cos } 10t \qquad \text{(h)}$$

Graphic representation of x(t) in Equation (h) is similar to the graph on the right with amplitudes increase rapidly with time t.

Physically, the amplitudes x(t) represents the elongation of the attached elastic pad on which the base of the stamping machine is attached.



(d) Determine the time to break the elastic support pad:

Since the elastic pad is expected to break at an elongation of 0.03 m, we may determine the time to reach this elongation (t_f) by the following mathematical expression in Equation (h):

$$0.03=0.005\text{Cos}10t_f\ + \frac{1}{200}Sin10t_f\ - \frac{t_f}{20}\text{cos}10t_f = \left(0.005 - \frac{t_f}{20}\right)\text{Cos}10t_f + \frac{1}{200}\text{Sin}10t_f$$

Solving for t_f from the above equation leads to t_f = 0.7 sec from the beginning of the resonant vibration. A more accurate solution of t_f = 0.862s was obtained by using Newton-Raphson method as will be presented in Section 10.3.2 in Chapter 10.

Example 8.10 (p.270)

A vehicle supported by suspension systems involving coilovers* as illustrated in Figure 8.21. The vehicle is rolling over a rough road with wavy road surface that could be described by a sine function, which results in a cyclic force of F(t) = 1200 Sin(10t) acting on the vehicle.

(* A coilover is a vibration isolator with a damper inside a circular coil spring).

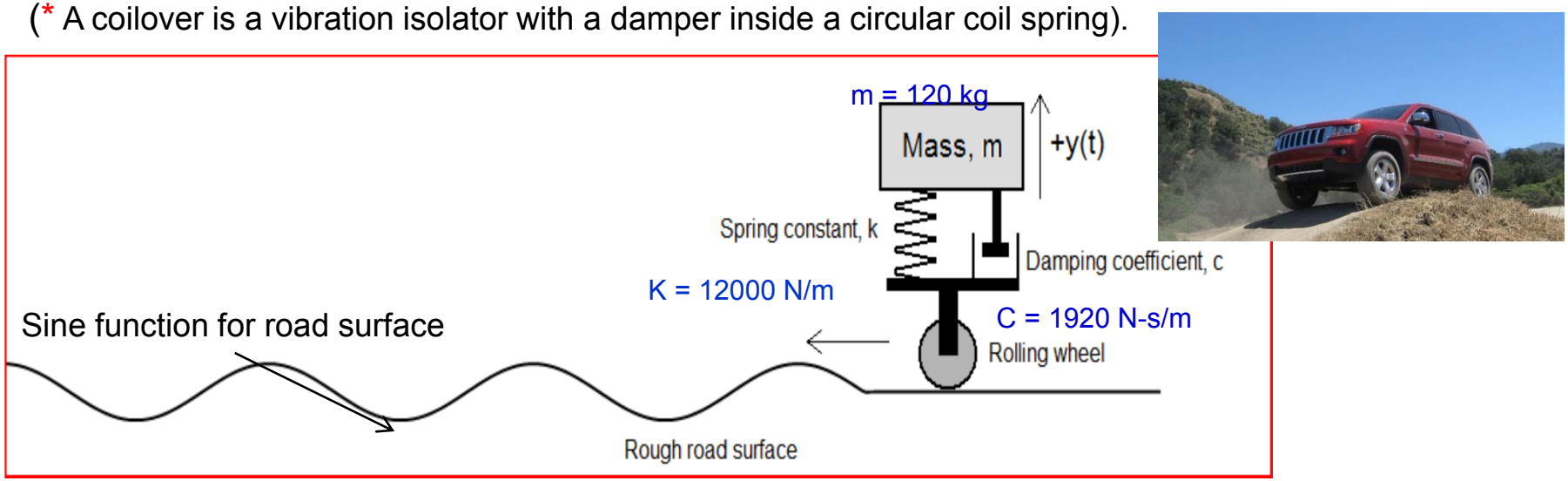


Figure 8.21 A vehicle travelling on a rough road surface

Response to the following questions:

(a) The appropriate differential equation for the amplitude of vibration of the vehicle mass y(t).

(b) Solve the differential equation with graphical representation of the solution using the following initial conditions:

$$y(0) = 0 \quad \text{and} \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = 5m/s$$

(c) What will be the position of the vehicle mass 2 seconds after the initiation of the vibration?

(d) What will happen to the vehicle if the damper loses its function? Given your reason for your predicted consequences.

(e) Provide graphical representation of the amplitudes of the vibrating vehicle in the case (d) using the same initial conditions stipulated in (b).

Example 8.10 – Cont'd

Solution:

We observe the picture in Figure 8.21 that each vehicle suspension system supports a share of vehicle mass that is equal to 120 kg.The appropriate differential equation for the amplitude of the vibrating vehicle is expressed following Equation (8.31) to be:

$$120\frac{d^2y(t)}{dt^2} + 1920\frac{dy(t)}{dt} + 12000\,y(t) = 1200\,Sin10t \qquad (8.31)$$

or

$$\frac{d^2y(t)}{dt^2} + 16\frac{dy(t)}{dt} + 100\,y(t) = 10Sin10t \qquad (a)$$

with the initial conditions: y(0) = 0 and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 5m/s$ (b)

The solution of Equation (a) is obtained by following Equation (8.27) in the form:

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ = complementary solution and $y_p(t)$ = particular solution

We obtained the complementary from the homogeneous portion of Equation (a) to be:

$$y_h(t) = e^{-8t}\left(c_1\,Cos6t + c_2\,Sin6t\right) \qquad (c)$$

The particular solution $y_p(t)$ can be obtained by assuming a form of:

$$y_p(t) = A_1\,Cos\,10t + A_2 Sin10t$$

in which A1 and A2 are constants to be determined by substituting the above assumed form into Equation (a), resulting in A_1 = -1/16 and A_2 = 0

Example 8.10 – Cont'd

Thus, the particular solution $y_p(t)$ is:

$$y_p(t) = -(\text{Cos}10t)/16 \qquad \text{(d)}$$

The general solution of Equation (a) is the summation of expressions in (c) and (d) to be:

$$y(t) = e^{-8t}\left(c_1\, Cos6t + c_2\, Sin6t\right) - \frac{1}{16} Cos10t \qquad \text{(e)}$$

In which the two constants c_1 and c_2 are determined from the specified conditions in Equation (b) to be: c_1 = 1/16 and c_2 = 11/12, leading to the solution of Equation (a) to be:

$$y(t) = e^{-8t}\left(\frac{1}{16} Cos6t + \frac{11}{12} Sin6t\right) - \frac{1}{16} Cos10t \qquad \text{(f)}$$

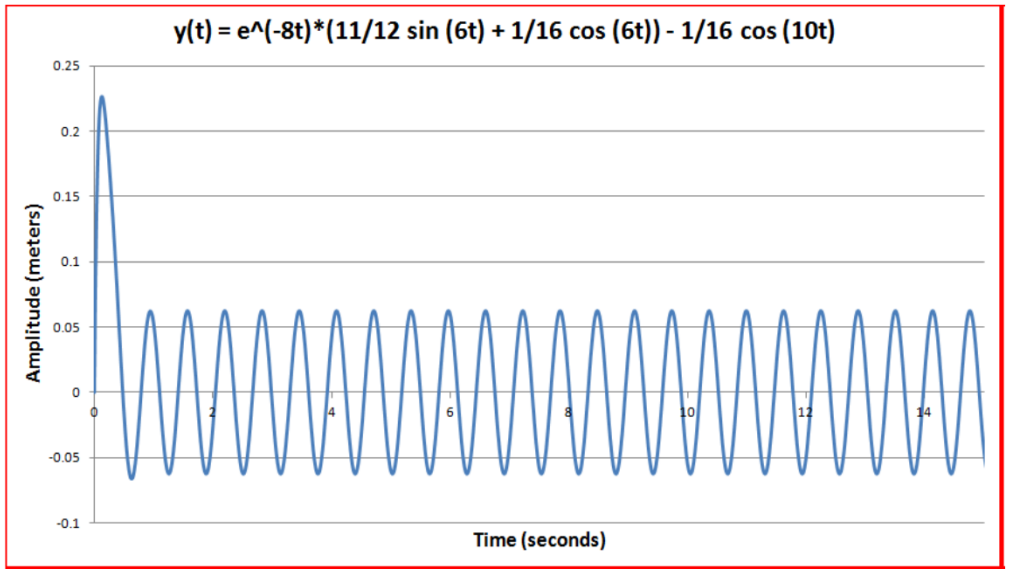


Figure 8.22 Graphical representation of solution in Equation (f)

We notice the vehicle had a maximum amplitude of vibration at around 0.25 s after hitting the curved road surface, but this excessive amplitude was "damped down" shortly afterward by the dampers inside the coilover

Example 8.10 – Cont'd

(c) The amplitude of vibration of the vehicle 2 seconds into the vibration may be solved by letting t = 2 seconds in Equation (f) resulting in y(2) = Cos 20 (radian)/16 = 0.025 m or 0.25 cm.

(d) Consequence in the case the damper in the coilover ceases to function:

In such case, the vehicle would be supported by the spring in the suspension system only. Mathematically, it would mean the damping coefficient c = 0, and hence the disappearance of the term dy(t)/dt in Equation (a). Consequently, Equation (a) will have a form:

$$\frac{d^2(t)}{dt^2} = \omega_o^2 y(t) = 10\,Sin\omega t \qquad \text{(g)}$$

in Equation (g). The natural frequency of the mass-spring structure is ω_o = 10 rad/s; whereas the frequency of the applied force, also is ω = 10 rad/s. We have situation of resonant vibration of the vehicle because $\omega_o = \omega$ = 10 rad/s.

The solution of Equation (g) is obtained by the usual procedure to be: $y(t) = y_h(t) + y_p(t)$, with:

$$y_h(t) = c_1\,Cos\omega_o t + c_2\,Sin\omega_o t \qquad \text{(h)}$$

The particular solution for this case of resonant vibration will have the particular solution: $y_p(t) = t(A_1 \cos\omega t + A_2 \sin\omega t)$ where the constant coefficients A_1 = -1/2 and A_2 = 0. We thus obtain the solution of Equation (g) to be:

$$y(t) = \frac{11}{20} Sin10t - \frac{t}{2} Cos10t \qquad \text{(j)}$$

after using the same initial conditions in Equation (b) in determining the constants c_1 and c_2 in Equation (h)

Example 8.10 – Cont'd

(e) Graphic representation of the solution in Equation (j) in the case in which the dampers inside the coilovers cease to function – the vehicle is now entirely supported by coil springs only.

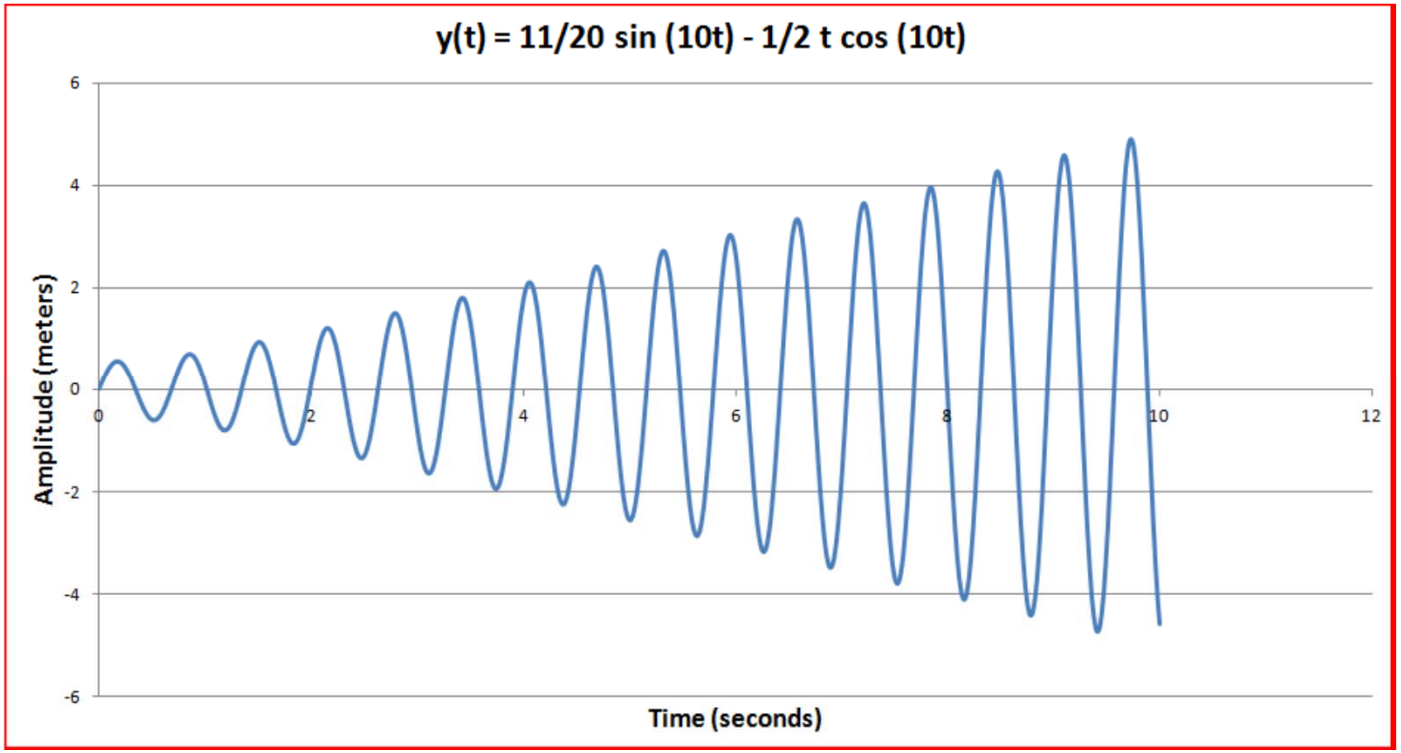


Figure 8.23 Fast rising of amplitudes of a vibrating vehicle in resonant vibration

We notice a very significant difference in the amplitude y(t) vs time t in both cases by the above graphic representation for the vehicle to vibrate in RESONANT VIBRATION with its magnitudes of vibration rising from zero at t=0 to almost 5 m in 10 seconds!! (the spring that supports the vehicle would have broken long before reaching that time of 10 seconds!!

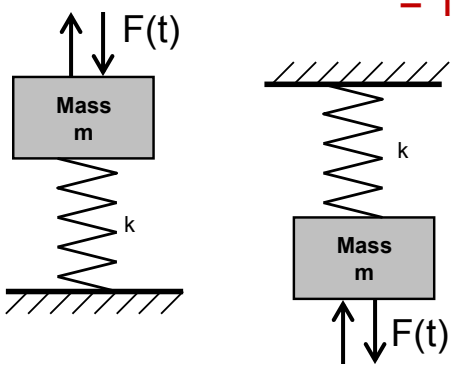
The sharp difference in the variation of the magnitudes of vibration with the presence of damper is Illustrated in the previous case (see the last slide as in Figure 8.22) in which the maximum amplitudes never exceed 0.0.6 m after about 0.4 second while the damper remains functioning.

8.8 **Near Resonant Vibration Analysis** (p.273)

We have learned from the previous section that resonant vibration of mass occurs when:

The frequency of the applies cyclic force to the mass **(ω)**

= The inhabitant natural frequency of the mass-spring system **(ω_o)**



There is little we can do to control the frequency of the applied cyclic force to the mass (ω) by the users (or our customers); but There are occasions in which the user would apply forces with $\omega \neq \omega_0$, but have $\omega \approx \omega_0$

Such is the case that we call "**Near Resonant**" vibration

Because we have the case of $\omega \neq \omega_0$ derived before Section 8.7 for the case $F(t)=F_0 Cos\ \omega t$, we could use the solution obtained for that case for the present case:

$$y(t) = c_1 Cos\omega_0 t + c_2 Sin\omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} Cos\omega t \qquad (8.35)$$

If we impose the same initial conditions:

y(0) = 0 for initial displacement, and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$ for initial velocity,

we would have the arbitrary constants in Equation (8.35) determined to be:

$$c_1 = -\frac{F_0}{m(\omega_o^2 - \omega^2)} \quad \text{and} \quad c_2 = 0$$

The complete solution for the differential equation in Equation (8.35) becomes:

$$y(t) = \frac{F_o}{M(\omega_o^2 - \omega^2)}\left[Cos(\omega t) - Cos(\omega_o t)\right] \qquad (8.37)$$

8.8 Near Resonant Vibration Analysis – Cont'd

By using the expressions for "half-angles" in trigonometry typically learned in high-school math course, the Subtractions of two cosine functions in Equation (8.37) may be expressed in the following form:

$$Cos\alpha - Cos\beta = -2Sin\frac{1}{2}(\alpha+\beta)Sin\frac{1}{2}(\alpha-\beta)$$

Substituting the above relation into Equation (8.37) will lead to the following expression:

$$y(t) = \frac{2F_o}{M(\omega_o^2 - \omega^2)} Sin\left[(\omega_o + \omega)\frac{t}{2}\right] Sin\left[(\omega_o - \omega)\frac{t}{2}\right] \qquad (8.38)$$

Because we have $\omega \approx \omega_0$ to be the condition of near-resonant vibration, hence $\omega_0 - \omega \rightarrow 0$ in Equation (8.38), we may make the following approximation and derive two special relationships:

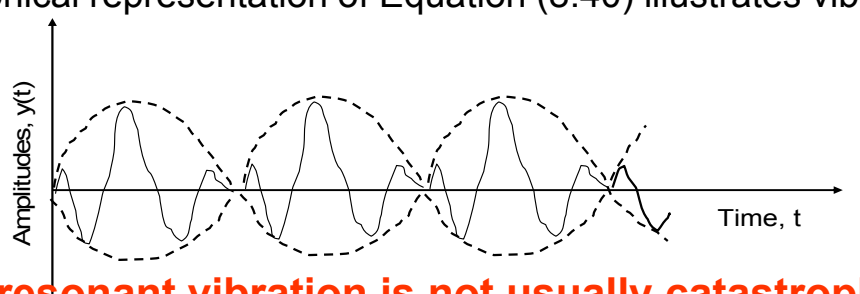
$$\frac{\omega_o + \omega}{2} \approx \omega \quad and \quad \frac{\omega_o - \omega}{2} = \varepsilon \qquad (8.39a,b)$$

in which the circular frequency $\varepsilon << \omega$ (the frequency of the exciting force)

Consequently, the solution in Equation (8.38) can be expressed as:

$$y(t) = \left[\frac{2F_o}{M(\omega_o^2 - \omega^2)}\right] Sin(\varepsilon t)\, Sin(\omega t) \qquad (8.40)$$

Graphical representation of Equation (8.40) illustrates vibration in oscillations with "beats", with:



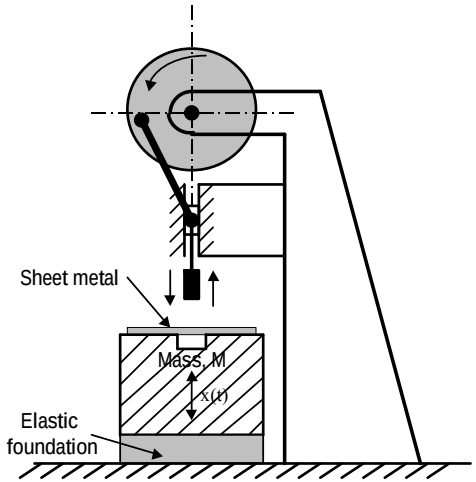
$f_b = \dfrac{\varepsilon}{2\pi}$ to be the frequency of the beats, and

$y(t) = \left[\dfrac{2F_o}{M(\omega_o^2 - \omega^2)}\right]$ to be the maximum amplitudes

<u>Near resonant vibration</u> is <u>not</u> usually catastrophic to the structure as does by resonant vibration, but it can cause unwanted disturbances and often leads to structure failures of structures due to FATIGUE of the material.

Example 8.11 (p.275):

The stamping machine in Example 8.9 is used to produce shallow metal cups from flat aluminum sheet metals.



The bench that supports the flat sheet metal has a mass of 1000 kg and it is bolted to an elastic foundation that has an equivalent spring constant k = 25000 N/m.

A measurement of the stamping force indicated a force function: F(t) = 1000 Sin(4.95t) applied to the bench with mass M during the stamping process.

Determine the following:

(a) Derive the differential equation and the appropriate conditions that describe the instantaneous position x(t) of the support bench,
(b) Solve the differential equation for the amplitudes of the vibrating bench x(t),
(c) Graphically illustrate the amplitudes of the vibrating machine vs. time,
(d) The maximum deflection of the elastic foundation to which the bench is attached,
(e) Would the elastic foundation break if its maximum allowed elongation is 5 cm
(f) Estimate the time required to break the elastic foundation should it happen.

Example 8.11 – Cont'd**:**

Solution:

(a) According to Equation (8.31), we have the following differential equation with appropriate conditions for the present problem:

$$1000\frac{d^2x(t)}{dt^2} + 25000\,x(t) = 1000\,Sin(4.95t) \qquad \text{(a)}$$

with given initial conditions:

$$x(0) = 0 \quad \text{and} \quad \left.\frac{dx(t)}{dt}\right|_{t=0} = 0 \qquad \text{(b)}$$

(b) Solution for Equation (a) with the specified conditions in Equation (b) is:

$$x(t) = -4.02\,\text{Cos}(4.95t)\,\text{Sin}(0.025t) \qquad \text{(c)}$$

(c) Graphical representation of the solution x(t) in Equation (c) obtained by MatLAB software is shown in Figure 8.27:

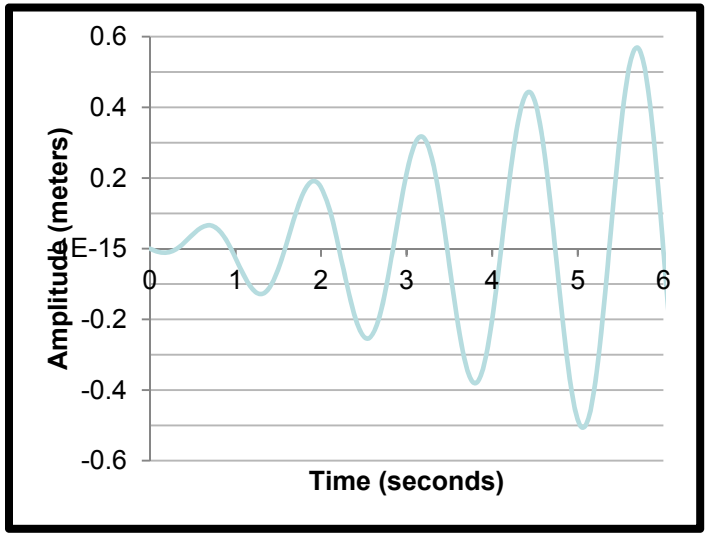


Figure 8.27 Amplitude of Near-Resonant vibration of the bench in the first 6 seconds:

Example 8.11 – Cont'd**:**

(c) Graphical representation of the solution x(t) in Equation (c) – obtained by MatLAB software

Fig. 8.28 Beats within one cycle for the near-resonant vibration of the support bench

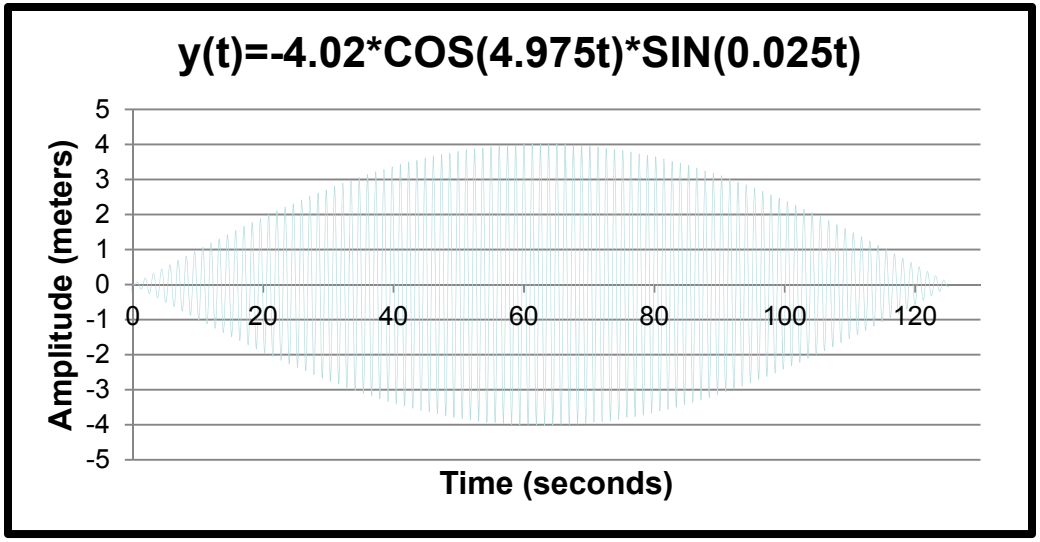
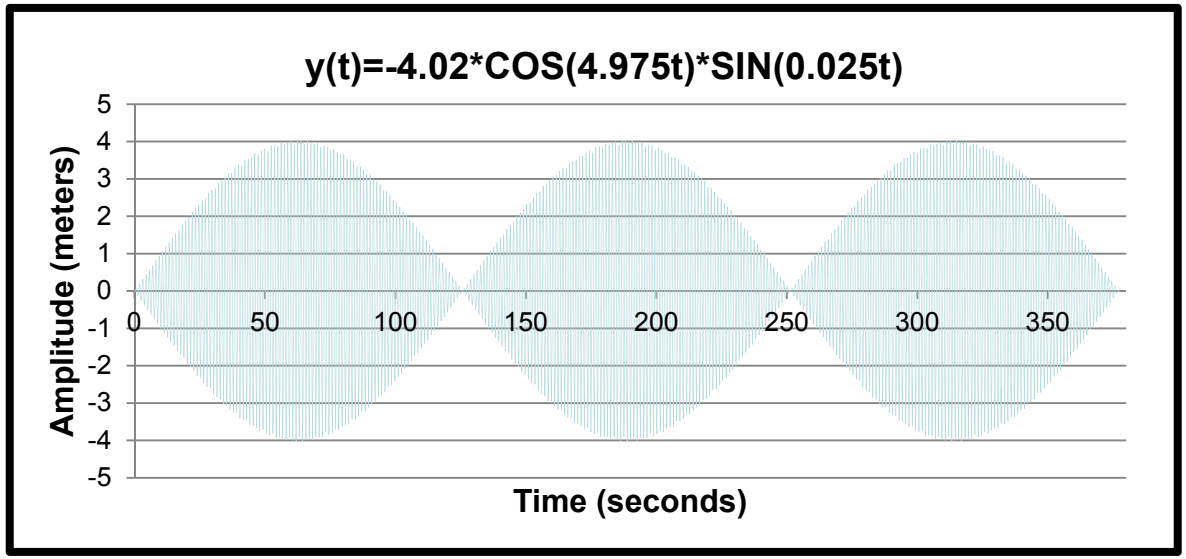


Fig. 8.29 Near-resonant vibration of the support bench with "beats" in each of these 3 cycles



Example 8.11 – Cont'd**:**

(d) The maximum deflection of the vibrating machine bench, and thus the maximum elongation of the elastic pad of the support foundation is 4.02 m as shown in Figures 8.28 and 8.29.

.

(e) Because the maximum elongation of the elastic foundation at 4.02 m has grossly exceeded the maximum allowed elongation of 0.05 m, so the elastic foundation will break.

(f) The time at which the elastic foundation breaks appears to be 0.5 seconds as indicated in Figure 8.27.

8.9 **Natural Frequencies of Structures and Modal Analysis** (p.277)

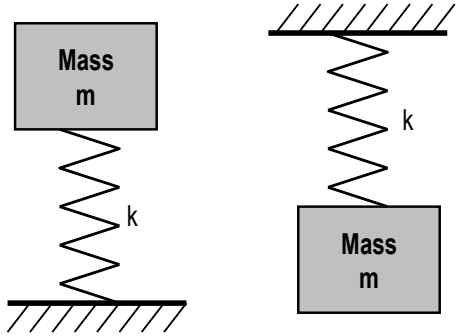
We have learned in this chapter that simple structures such as a mass attached to a spring are vulnerable to **resonant vibration** if the frequency of the applied cyclic force (**ω**) coincides with the natural frequency of the mass-spring structure (**ω_0**).

The consequence of structures, including the simple mass-spring system in resonant vibration is very serious with likely immediate structural failure due to rapid amplification of the magnitudes of vibration of the structure in such situation.

To avoid such happening, we need to know the *natural frequency of the structure*, so that we can avoid resonant vibration from happening to the structure by **not** applying any cyclic forces to the structure at frequencies that coincide with any natural frequencies of the structure

Natural frequency of the structure *(ω_n, with n = 1,2,3,….,n):*

We have learned from Section 8.4 that the natural frequency of a simple mass-spring system has a natural frequency $\omega_0=\sqrt{k/m}$

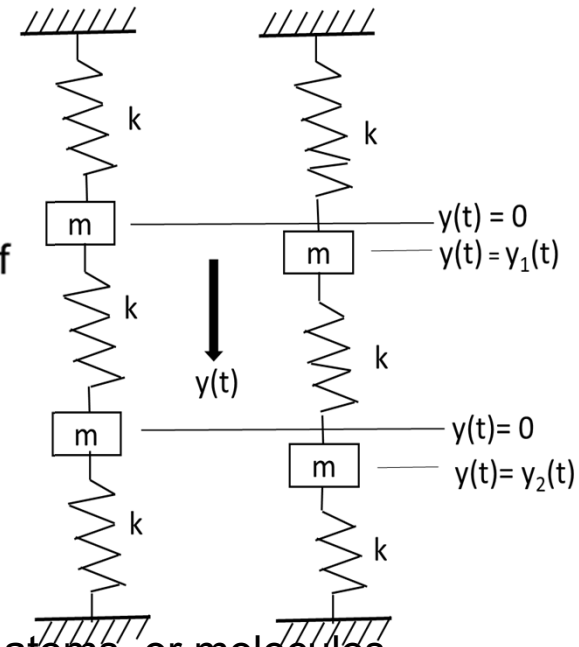


In general, a single mass supported by a single spring has ONE natural frequency, and the number of natural frequencies of solid structures increases by the increase of number of masses interconnected by many more elastic bonds.

8.9 Natural Frequencies of Structures and The Modal Analysis – Cont'd

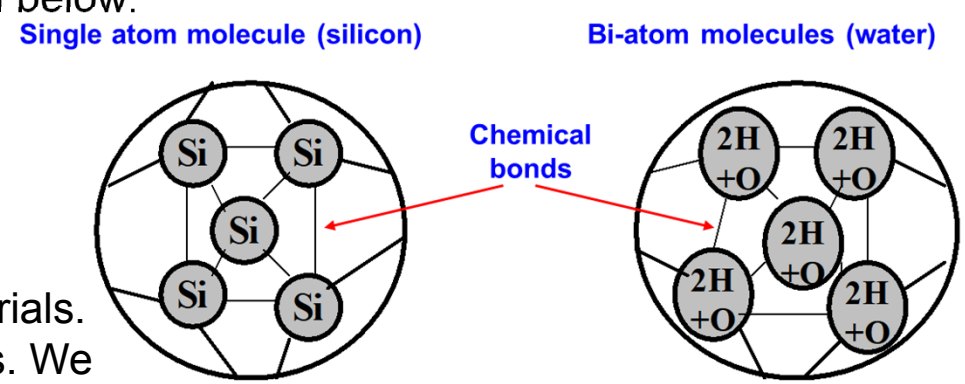
Natural frequency of the structure *(ω_n, with n = 1,2,3,….,n) – Cont'd:*

Also, we have learned from Section 4.8.3 (p.146) that there are TWO natural frequencies for a structure that involve TWO solid masses in free vibration, with $\omega_1 = +\sqrt{\lambda}$ and $\omega_2 = -\sqrt{\lambda}$ from Equation (4.52) in which λ is the eigenvalue of the amplitude of vibration of these masses.



We may thus observe that a structure with TWO solid masses supported by springs would have TWO natural frequencies: ω_1 and ω_2, or ω_n with n = 1,2.

Now, let us imagine structures made of "real materials" with atoms, or molecules supported by "molecular bonds," as illustrated below:



We realize that by nature, some molecules are made by single atoms and some others involve multiple kinds of atoms, and there are zillions such atoms or molecules interconnected by chemical bonds in all materials. These bonds may be treated as elastic bonds. We may thus imagine there are zillions numbers of natural frequencies of structures made of real materials that engineers use in reality.

8.9 Natural Frequencies of Structures and The Modal Analysis – Cont'd

Natural frequency of the structure *(ω_n, with n = 1,2,3,….,n) – Cont'd:*

We will further recognize a fact that deformation of solids results elongations or contractions of the chemical bonds between the atoms or molecules, which produce molecular forces as illustrated in Figure 8.30::

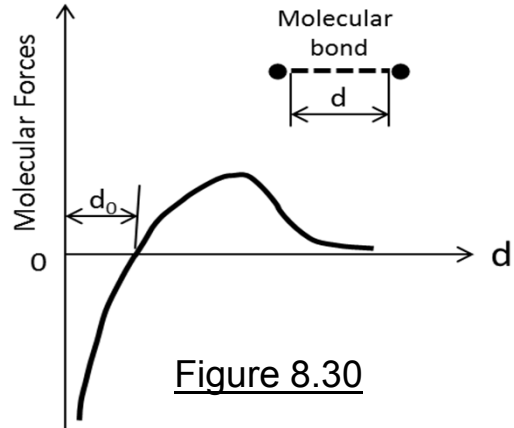


Figure 8.30

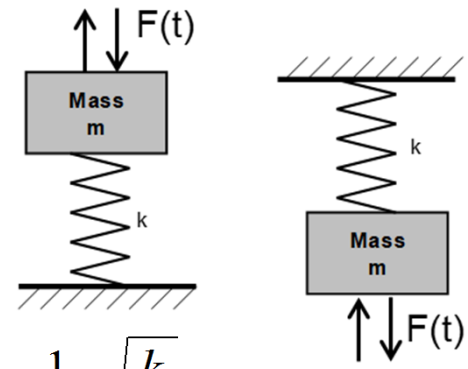
These molecular forces in both forms of repulsions and attractions with their magnitudes varying according to the distance between their terminals, which exhibit similar ways as the spring forces that cause the attached masses to vibrate from their equilibrium positions in simple mass-spring systems illustrated in previous cases.

Now if we imagine the structure of preferred geometry and sizes that involve zillion number of atoms or molecules (remember the well-known Avogadro's number defined to be the number of molecules contained in 1 mole of any gas or substance to be 6.022×10^{23}?), an elastic deformation of these molecular bonds could indeed prompt a "million" ways for their attached atoms or molecules to the free-vibration modes. It is thus not hard for one to imagine that there are, in theory, millions number of modes for the structure to vibrate, and the number of modes of vibration of structure is thus "of MANY, and we may express the "natural frequencies" associate with each of these modes of vibration to be: ω_n *with* $n = 1,2,3,\ldots,n$ where n designates the mode number.

MODAL ANALYSIS (p.279)

MODAL ANALYSIS is a process of determining the natural frequency or frequencies of a machine or structure .

For simple mass-spring systems with the mass being attached or supported by a single spring, the mass vibrates in one-degree-of freedom (because the motion of the mass is prompted by a single spring force)



One degree-of-freedom system has only ONE MODE of natural frequency – one natural frequency, ω_0 with

$$f = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (8.17)$$

For structures of complex geometry subjected to complex loading, there exists an infinite (∞) degree-of-freedom, and thus infinite number of natural frequencies – calling Mode 1, 2, 3, ……..natural frequencies, expressed by: ω_n: ω_1, ω_2, ω_3, .ω_∞

$$\omega_n = \sqrt{\frac{[k]}{[m]}}$$ with mode number n = 1, 2, 3, ….,n

where [k] and [m] are respective "stiffness matrix" and "mass matrix" of the structure. These matrices are obtained by numerical analyses, such as finite element stress analysis described in Chapter 11.

Every effort should be made not to apply any intermittent cyclic forces with frequency coinciding ANY of the natural frequency in any mode of the structure